

# DECEPTIVE CONVERGENCE OF FOURIER SERIES ON $SU(2)$

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**1. Introduction and summary.** In this paper  $G$  will always denote a compact group that has a faithful finite-dimensional unitary representation. The Haar measure on  $G$  will be denoted by  $\mu$ , and we normalize  $\mu$  so that  $\mu(G)=1$ . By a *representation* of  $G$  we will mean a finite-dimensional unitary representation of  $G$ . Let  $U$  be a faithful representation of  $G$  and let  $V=U \oplus \bar{U}$  be the direct sum of  $U$  and its complex conjugate representation. Let  $T_0(U)$  be the space of constant functions on  $G$ , let  $T_1(U)$  be the linear space spanned by  $T_0(U)$  and the coordinate functions of  $V$ , and for each integer  $n \geq 1$  let  $T_n(U)$  be the space spanned by all functions  $f_1 f_2 \cdots f_n$  where  $f_i \in T_1(U)$  for  $1 \leq i \leq n$ . We will call  $T_n(U)$  the *space of  $U$ -trigonometric polynomials of degree  $\leq n$* . For each nonnegative integer  $n$  let  $U_n$  be the orthogonal projection of  $L^2(G)$  onto  $T_n(U)$ . There is a natural way to extend  $U_n$  to a projection from  $L^1(G)$  onto  $T_n(U)$  (see (2.1)) and we will denote this extension by  $U_n$ . If  $f$  is a function in  $L^1(G)$  we will call the sequence  $\{U_n f\}$  the  *$U$ -Fourier series for  $f$* , and we will call  $U_n f$  the  *$n$ th partial sum of the  $U$ -Fourier series for  $f$* . If  $f \in L^2(G, \mu)$ , then  $U_n f \rightarrow f$  in  $L^2(G, \mu)$  as  $n \rightarrow \infty$ . If  $x \in G$  then the  *$U$ -Fourier series for  $f$  at  $x$*  is defined to be the sequence  $\{U_n f(x)\}$ .

If  $G=T$  is the group of complex numbers of absolute value 1, and  $U$  is the 1-dimensional representation of  $T$  on  $C$  defined by

$$(1.1) \quad U(e^{it})z = e^{it}z \quad \text{for all } e^{it} \in T, z \in C$$

then the  $U$ -trigonometric polynomials of degree  $\leq n$  are ordinary trigonometric polynomials of degree  $\leq n$ , and for any  $f \in L^1(T)$ ,  $U_n f$  is the  $n$ th partial sum of the ordinary Fourier series for  $f$ .

Let  $\{a_n\}, \{b_n\}$  be two sequences of complex numbers. We say that  $\{a_n\}$  and  $\{b_n\}$  *co-converge* if they either both converge to the same limit or both diverge. Let  $U$  and  $W$  be two faithful representations of the compact group  $G$ . We will say that  $U$  and  $W$  are *series equivalent* if for every  $f$  in  $L^1(G)$  and every  $x$  in  $G$  the sequences  $\{U_n f(x)\}$  and  $\{W_n f(x)\}$  co-converge. Series equivalence is clearly an equivalence relation on the set of faithful representations of  $G$ . It is easy to show that any two faithful representations of  $T$  are series equivalent (see 2.2). For the group  $SU(2)$  of  $2 \times 2$  unitary matrices with determinant 1 the situation is more complicated.  $SU(2)$  has exactly one irreducible representation  $R^n$  of dimension  $n$  for each

positive integer  $n$  (see [8, p. 137]). It follows from (3.19) and (7.18) that the representations  $R^q \oplus R^{p+1}$  where  $1 \leq q \leq p$  and  $p \equiv q \pmod{2}$  form a complete set of equivalence class representatives for the relation of series equivalence on the faithful representations of  $SU(2)$ . If  $U$  is a faithful representation of  $SU(2)$  which is series equivalent to  $R^q \oplus R^{p+1}$  we will say that  $U$  is of type  $(q, p)$ .

Let  $U$  be a faithful representation of  $G$ , let  $x \in G$  and let  $f$  be a function in  $L^1(G)$  which is continuous at  $x$ . If the  $U$ -Fourier series for  $f$  at  $x$  converges to a value different from  $f(x)$  we will say that the  $U$ -Fourier series for  $f$  converges *deceptively* at  $x$ . It is well known that the ordinary Fourier series of a function in  $L^1(T)$  cannot converge deceptively [10, p. 89] and it follows from [5, p. 683] that the  $R^2$ -Fourier series of a function in  $L^1(SU(2))$  cannot converge deceptively. Let  $S$  be a subset of  $L^1(G)$ . If no function  $f$  in  $S$  has a point of continuity at which the  $U$ -Fourier series for  $f$  converges deceptively, we will say that  *$U$ -Fourier series are honest for functions in  $S$* .

In this paper we will prove the following results. Let  $S$  be the set of all functions in  $L^\infty(SU(2))$  which are continuous except on a set of Hausdorff dimension  $\leq 2$ , and let  $T$  be the set consisting of those functions in  $S$  whose set of discontinuities has Hausdorff dimension  $< 2$ . Let  $U$  be a faithful representation of  $SU(2)$ . Then  $U$ -Fourier series are honest for functions in  $T$ ;  $U$ -Fourier series are honest for functions in  $S$  if and only if  $U$  is of type  $(1, 1)$ ,  $(2, 2)$  or  $(3, 3)$ ;  $U$ -Fourier series are honest for functions in  $L^\infty(SU(2))$  if and only if  $U$  is of type  $(1, 1)$ ,  $(2, 2)$  or  $(3, 3)$ ; and  $U$ -Fourier series are honest for functions in  $L^1(SU(2))$  if and only if  $U$  is of type  $(1, 1)$ . With each faithful representation  $U$  of  $SU(2)$  we can associate a set  $D_U$  (the *deception set* of  $U$ ) with the following two properties.

I. If  $x \in SU(2)$  and  $y$  is any element of  $x D_U$  different from  $x$ , then there exists a function  $f$  in  $L^1(SU(2))$  such that  $f$  is analytic except at  $y$  and the  $U$ -Fourier series for  $f$  converges deceptively at  $x$ .

II. If  $x \in SU(2)$  and  $f$  is any function in  $L^1(SU(2))$  which is continuous on  $x D_U$  then the  $U$ -Fourier series for  $f$  does not converge deceptively at  $x$ .

In (7.1) we give a very explicit description of  $D_U$  which shows that if  $U$  is of type  $(q, p)$  with  $p$  odd, then  $D_U$  consists of all elements of  $SU(2)$  whose eigenvalues are  $p$ th roots of unity. The main tool for proving the above results is the technical Theorem 5.63 which gives an explicit formula for the error  $f(x) - \lim_{n \rightarrow \infty} U_n f(x)$ , valid for a fairly large class of functions. Let  $U$  be any faithful representation of  $SU(2)$ , and let  $f$  be a function in  $L^2(SU(2))$ . Then the set of points where the  $U$ -Fourier series for  $f$  converges deceptively has measure zero. It would be desirable to prove this result for arbitrary functions in  $L^1(SU(2))$ , but I have been unable to do this.

**2. Series equivalence of the representations of  $T$ .** Let  $U$  be a faithful representation of the compact group  $G$ . Since  $T_1(U)$  is the space spanned by the coordinate functions of a representation of  $G$ ,  $T_1(U)$  is clearly left and right translation in-

variant, and hence  $T_n(U)$  is a left and right translation invariant subspace of  $L^2(G)$  for all  $n$ . Also  $T_n(U)$  is closed (in fact finite dimensional) in  $L^2(G)$ , so  $T_n(U)$  is a two-sided ideal in  $L^2(G)$  by [4, §31F and 39E]. By the structure theory for such ideals [4, §39] we know that  $T_n(U)$  contains a unique central idempotent  $D_n^U$  with the property that  $f \rightarrow f * D_n^U$  is the projection of  $L^2(G)$  onto  $T_n(U)$ , i.e.

$$(2.1) \quad U_n f = f * D_n^U \quad \text{for all } f \in L^2(G).$$

(The  $*$  here denotes convolution.) We will call the sequence  $\{D_n^U\}$  ( $n=0, 1, 2, \dots$ ) the *Dirichlet kernel* for  $U$ . The right-hand side of (2.1) makes sense for any  $f$  in  $L^1(G)$  and we use (2.1) to define  $U_n f$  for any  $f$  in  $L^1(G)$ .

**PROPOSITION 2.2.** *Any two faithful representations of  $T$  are series equivalent.*

**Proof.** Let  $W$  be any faithful representation of  $T$ , and let  $U$  be the representation defined in (1.1). Note that for each nonnegative integer  $n$ ,  $T_n(U)$  is the linear space spanned by  $\{e^{ikt} : -n \leq k \leq n\}$ . If  $f$  is a function in  $L^1(T)$  whose (ordinary) Fourier series is  $f \sim \sum c_k e^{ikt}$  then the  $U$ -Fourier series for  $f$  is  $\{\sum_{k=-n}^n c_k e^{ikt}\}$ , and the  $W$ -Fourier series for  $f$  is  $\{\sum_{k \in A(n)} c_k e^{ikt}\}$  where  $A(n) = \{k \in \mathbb{Z} : e^{ikt} \in T_n(W)\}$ . We will show that  $W$  and  $U$  are series equivalent.

Let  $\chi$  be the character of  $W$ . We can write  $\chi(e^{it}) = \sum a_n e^{int}$  where the  $a_n$  are non-negative integers and all but a finite number of the  $a_n$ 's are zero. Let  $N$  be the largest integer such that  $a_N + a_{-N} > 0$ . For any positive integer  $k$  we have

$$(2.3) \quad T_k(W) \subseteq T_{Nk}(U).$$

Since  $W$  is a faithful representation of  $T$  it follows from [2, pp. 189–190] that the algebra generated by  $\{e^{int} : a_n > 0\} \cup \{e^{-int} : a_n > 0\}$  is the algebra of all trigonometric polynomials, and hence there exists an integer  $p$  such that  $T_N(U) \subseteq T_p(W)$ . Since

$$e^{iNt} T_{pN}(U) + e^{-iNt} T_{pN}(U) = T_{(p+1)N}(U)$$

we can show by induction that

$$(2.4) \quad T_{N(q+1)}(U) \subseteq T_{p+q}(W), \quad q \geq 0.$$

Combining (2.3) and (2.4) we obtain

$$(2.5) \quad T_{Nk - N(p-1)}(U) \subseteq T_k(W) \subseteq T_{Nk}(U), \quad k \geq p.$$

Let  $B(k) = \{n \in \mathbb{Z} : e^{int} \in T_{Nk}(U), e^{int} \notin T_k(W)\}$ . Then  $U_{Nk}f - W_kf = \sum_{n \in B(k)} c_n e^{int}$  and it follows from (2.5) that

$$\|U_{Nk}f - W_kf\|_\infty \leq 2N(p-1) \sup \{|c_j| : |j| \geq N(k-p+1)\}$$

whenever  $k \geq p$ . Thus by the Riemann-Lebesgue lemma we see that  $U_{Nk}f - W_kf$  converges uniformly to 0 on  $T$  as  $k \rightarrow \infty$ , and for any  $e^{ix} \in T$  we see that  $\{U_{Nk}f(e^{ix})\}$  and  $\{W_kf(e^{ix})\}$  co-converge. Also the Riemann-Lebesgue lemma shows that

$\{U_{Nk}f(e^{ix})\}$  and  $\{U_k f(e^{ix})\}$  co-converge, and it follows that  $\{W_k f(e^{ix})\}$  and  $\{U_k f(e^{ix})\}$  co-converge. Thus  $U$  and  $W$  are series equivalent.

**3. Formulas for the Dirichlet kernels.** Let  $G$  be a compact group and let  $\{\chi_a\}$  ( $a \in A$ ) be the set of all irreducible characters of  $G$ . If  $\phi$  is any character of  $G$  and  $a \in A$ , we will say that  $\chi_a$  is contained in  $\phi$  if  $(\chi_a, \phi) > 0$ . Here  $(\chi_a, \phi)$  denotes the inner product of  $\chi_a$  and  $\phi$ . Let  $\phi, \psi$  be two characters of  $G$ . We will write  $\phi > \psi$  if every irreducible character contained in  $\psi$  is also contained in  $\phi$ , and we will write  $\psi \sim \phi$  if  $\phi > \psi$  and  $\psi > \phi$ . Thus  $\phi \sim \psi$  if and only if the representations corresponding to  $\phi$  and  $\psi$  are quasi equivalent in the sense of [7, p. 631]. We will use the facts that

$$(3.1) \quad \phi < \psi \Rightarrow \phi + \psi \sim \psi,$$

$$(3.2) \quad \phi < \psi \text{ and } \phi_1 < \psi_1 \Rightarrow \phi\phi_1 < \psi\psi_1,$$

and similar obvious properties of the relations  $<$  and  $\sim$  without specifically mentioning them. Let  $E(\chi_a)$  be the two-sided ideal in  $L^2(G)$  generated by  $\chi_a$ . Then  $\{E(\chi_a) : a \in A\}$  is the set of all minimal two-sided ideals in  $L^2(G)$ , and the generating idempotent for  $E(\chi_a)$  is  $d_a \chi_a$  where  $d_a = \chi_a(e)$ . Let  $U$  be a faithful representation of  $G$ . Then since  $T_n(U)$  is a closed ideal in  $L^2(G)$  we have by [4, §39] that

$$T_n(U) = \sum E(\chi_a) \quad (\chi_a \in T_n(U))$$

and hence the Dirichlet kernel  $\{D_n^U\}$  for  $U$  is given by

$$(3.3) \quad D_n^U = \sum d_a \chi_a \quad (\chi_a \in T_n(U)).$$

Let  $\chi_U$  be the character of  $U$ . Then it follows from [5, p. 684] that  $\chi_a \in T_n(U)$  if and only if  $((1 + \chi_U + \bar{\chi}_U)^n, \chi_a) > 0$ , and hence (3.3) becomes

$$(3.4) \quad D_n^U = \sum d_a \chi_a, \quad \chi_a < (1 + \chi_U + \bar{\chi}_U)^n.$$

Let  $\theta$  be the function on  $SU(2)$  defined by

$$(3.5) \quad \theta(g) = \arccos \left( \frac{1}{2} \text{Trace}(g) \right), \quad g \in SU(2)$$

so that the eigenvalues of an element  $g$  of  $SU(2)$  are  $\exp(\pm i\theta(g))$ . For each positive integer  $n$ ,  $SU(2)$  has exactly one irreducible representation  $R^n$  of dimension  $n$ , and the character  $\chi_n$  of  $R^n$  is

$$(3.6) \quad \chi_n = \sin n\theta / \sin \theta$$

(see [8, p. 151]). Since  $\chi_{2n}(x) = 2n$  only if  $\theta(x) = 1$ , or equivalently only if  $x = e$ , the characters  $\chi_{2n}$  are all faithful (we call a character faithful if it is the character of a faithful representation). Note that  $\chi_{2n+1}(e) = \chi_{2n+1}(-e)$  so  $\chi_{2n+1}$  is never faithful. Let  $U$  be a faithful representation of  $SU(2)$  with character  $\chi_U = \sum n_k(U) \chi_k$ . Then  $\chi_U(-e) = \sum k n_k(U) (-1)^{k+1}$ . Since  $U$  is faithful  $\chi_U(-e) \neq \chi_U(e) = \sum k n_k(U)$ , and hence  $\chi_U$  contains some irreducible character of even dimension. Let  $p$  be the

greatest integer such that  $n_{p+1}(U) > 0$  (so  $p \geq 1$ ) and let  $q$  be the greatest integer such that  $q \equiv p \pmod{2}$  and  $n_q(U) > 0$  (if no such  $q$  exists then  $p$  is odd, and in this case we take  $q=1$ ). We will say that  $U$  is of type  $(q, p)$ . If  $q, p$  are any integers such that  $1 \leq q \leq p$  and  $q \equiv p \pmod{2}$  then  $R^q \oplus R^{p+1}$  is of type  $(q, p)$ . We will show in (7.18) that this definition of type agrees with the definition given in the introduction.

**THEOREM 3.7.** *Let  $U$  be a faithful representation of  $SU(2)$  of type  $(q, p)$ , and let  $\{D_n^U\}$  be the Dirichlet kernel for  $U$ . Then for every  $n \geq 3$*

$$(3.8) \quad D_n^U = \sum_{m=q+1}^{p+1} (-1)^{p+m+1} D_{(n-1)p+m},$$

where

$$(3.9) \quad D_j = \sum_{k=1}^j k \chi_k.$$

**Proof.** By the Clebsch-Gordan formula [8, p. 163] we know

$$(3.10) \quad \chi_r \chi_s = \sum_{j=1}^r \chi_{r+s+1-2j} \quad \text{if } r \leq s.$$

We define  $S(n)$  and  $S^*(n)$  by the formulas

$$(3.11) \quad \begin{aligned} S(n) &= \sum \chi_j & (1 \leq j \leq n) \\ S^*(n) &= \sum \chi_j & (1 \leq j \leq n, j \equiv n \pmod{2}). \end{aligned}$$

It then follows from (3.10) that

$$(3.12) \quad \begin{aligned} \chi_r \cdot S(n) &\sim S(n+r-1) & (r \leq n) \\ \chi_r \cdot S^*(n) &\sim S^*(n+r-1) & (r \leq n+1) \end{aligned}$$

and that

$$(3.13) \quad \begin{aligned} \chi_r \cdot S(n) &< \chi_s \cdot S(n) & (r \leq s \leq n) \\ \chi_r \cdot S^*(n) &< \chi_s \cdot S^*(n) & (r \leq s \leq n+1, r \equiv s \pmod{2}). \end{aligned}$$

**LEMMA 3.14.** *Let  $U$  be a faithful representation of  $SU(2)$  of type  $(q, p)$  and if  $q=1$  assume that  $\chi_1$  is contained in  $\chi_U$ . Then for every  $n \geq 3$*

$$(3.15) \quad (\chi_U)^n \sim S((n-1)p+q) + S^*(np+1)$$

where  $S(n)$  and  $S^*(n)$  are as defined in (3.11).

**Proof.** Using the Clebsch-Gordan formula (3.10) we get that  $(\chi_U)^3 \succ (\chi_{p+1})^3 \sim S^*(3p+1)$  and  $(\chi_U)^3 \succ \chi_{p+1}^2 \chi_q \sim S^*(2p+q)$  and hence

$$(3.16) \quad (\chi_U)^3 \succ S^*(3p+1) + S^*(2p+q) \sim S^*(3p+1) + S(2p+q),$$

since  $p$  and  $q$  have the same parity. Let us say that  $\chi_r$  is *larger* than  $\chi_s$  if and only if  $r > s$ . The largest irreducible character in  $(\chi_U)^3$  is the largest character in  $(\chi_{p+1})^3$

which is  $\chi_{3p+1}$ . The largest character in  $(\chi_U)^3$  whose dimension has parity different from  $3p+1$  is the largest character in  $\chi_{p+1}^2 \chi_q$  which is  $\chi_{2p+q}$ . Thus  $(\chi_U)^3 < S^*(3p+1) + S(2p+q)$ . This fact combined with (3.16) proves the lemma for the case  $n=3$ . Suppose now that we know that (3.15) holds for a given  $n \geq 3$ . Then by (3.12), (3.13), (3.1) and (3.2) we have

$$\begin{aligned} (\chi_U)^{n+1} &\sim \chi_U(S((n-1)p+q) + S^*(np+1)) \\ &\sim (\chi_q + \chi_{p+1})(S((n-1)p+q) + S^*(np+1)) \\ &\sim S(np+q) + S^*(np+q) + S^*((n+1)p+1). \end{aligned}$$

Since  $S^*(np+q) < S(np+q)$ , (3.15) also holds for  $(n+1)$ , and the lemma is proved.

It follows from Lemma 3.14 that for  $n \geq 3$

$$\begin{aligned} (1 + \chi_U)^n &\sim \sum_{j=0}^n (\chi_U)^j \sim S((n-1)p+q) + S^*(np+1) \\ (3.17) \quad &\sim S((n-1)p+q+1) + \sum_{m=1}^{(p-q)/2} \chi_{(n-1)p+q+1+2m}. \end{aligned}$$

It is easy to see that (3.17) holds even if  $q=1$  and  $\chi_1$  is not contained  $\chi_U$ , since  $\chi_1=1$ . We have  $D_j - D_{j-1} = j\chi_j$  from the definition (3.9). Since  $\chi_U$  is a real character it follows from (3.4) that

$$D_n^U = \sum k\chi_k, \quad \chi_k < (1 + \chi_U)^n,$$

and using this with (3.17) we get

$$\begin{aligned} (3.18) \quad D_n^U &= D_{(n-1)p+q+1} + \sum_{m=1}^{(p-q)/2} (D_{(n-1)p+q+1+2m} - D_{(n-1)p+q+2m}) \\ &= \sum_{m=q+1}^{p+1} (-1)^{m+p+1} D_{(n-1)p+m} \end{aligned}$$

for  $n \geq 3$ . This completes the proof of Theorem 3.7.

**COROLLARY 3.19.** *If  $U$  is a faithful representation of  $SU(2)$  of type  $(q, p)$  then  $U$  is series equivalent to  $R^q \oplus R^{p+1}$ .*

**COROLLARY 3.20.** *If  $U$  is a faithful representation of  $SU(2)$  of type  $(q, p)$  then  $\|D_n^U\|_\infty < 2n^3p^3$  for  $n \geq 3$ .*

**Proof.**

$$\begin{aligned} \|D_n^U\|_\infty &= D_n^U(e) \leq D_{np+1}(e) \\ &= \sum_{j=1}^{np+1} j^2 < 2n^3p^3 \quad \text{for } n \geq 3. \end{aligned}$$

**4. Some technical lemmas.** In this section all functions denoted by capital Latin letters will be complex valued measurable functions of period  $2\pi$  defined on  $\mathbb{R}$ . If  $F$  is a function we will write

$$(4.1) \quad \|F\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(t)| dt$$

whenever the integral on the right exists, and  $L^1$  will denote the space of functions  $F$  for which  $\|F\|_1$  is finite. If  $F, G$  are functions we will write

$$(4.2) \quad (F, G) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) \overline{G}(t) dt$$

whenever the integral on the right exists. If  $F$  has a right-hand (left-hand) limit at a point  $t_0$  we will denote this limit by  $F_+(t_0)$  ( $F_-(t_0)$ ).

This section contains a number of technical lemmas that will be used later in the paper. Since most of the lemmas are not interesting in themselves, I would suggest that the reader go to §5 at this point, and refer back to the material in this section as it is used.

For  $0 < r < 1$  define

$$(4.3) \quad A_r(t) = (1 - r^2)(1 - 2r \cos t + r^2)^{-1} = 1 + 2 \sum_{k=1}^{\infty} r^k \cos kt,$$

$$(4.4) \quad B_r(t) = -r^{-1} \sin t A'_r(t) = 1 + \sum_{k=1}^{\infty} ((k+1)r^k - (k-1)r^{k-2}) \cos kt.$$

$A_r$  and  $B_r$  are positive even functions,  $\|A_r\|_1 = \|B_r\|_1 = 1$  and for any  $\delta > 0$ ,  $A_r$  and  $B_r$  both converge uniformly to 0 on  $[\delta, 2\pi - \delta]$  as  $r \rightarrow 1$  (see [10, pp. 96, 97 and 100]). Hence it follows from [10, p. 86] that if  $F$  is any function in  $L^1$  such that  $F_+(0)$  and  $F_-(0)$  both exist, we have

$$(4.5) \quad \lim_{r \rightarrow 1} (F, A_r) = \lim_{r \rightarrow 1} (F, B_r) = \frac{1}{2}(F_+(0) + F_-(0)).$$

Define

$$(4.6) \quad \begin{aligned} C_r(t) &= (A_r(t) - r^2 B_r(t))(1+r)^{-1} \\ &= (1-r) \left[ 1 + \sum_{k=1}^{\infty} (k+1)r^k \cos kt \right]. \end{aligned}$$

**LEMMA 4.7.** *Let  $F$  be a function in  $L^1$  such that  $F$  is bounded on a neighborhood of 0 and such that the limit  $\lim_{r \rightarrow 1} (F, C_r) = L$  exists. Then  $L = 0$ .*

**Proof.** Since  $(F, C_r) = 0$  for any odd function  $F$ , we may assume without loss of generality that  $F$  is even. Suppose  $L \neq 0$ . Then we may assume without loss of generality that  $F$  is real valued, and  $L > 0$ . Choose  $\delta$  such that  $0 < \delta < \pi$  and such that  $F$  is bounded on  $[-\delta, \delta]$ . Let  $g$  be the characteristic function of  $[-\delta, \delta]$ , and

let  $G$  be the periodic function of period  $2\pi$  that agrees with  $Fg$  on  $[-\pi, \pi]$ . Then  $G$  is bounded, and

$$(4.8) \quad \lim_{r \rightarrow 1} (G, C_r) = \lim_{r \rightarrow 1} (F, C_r) + \lim_{r \rightarrow 1} \frac{(G-F, A_r)}{1+r} - \lim_{r \rightarrow 1} \frac{r^2(G-F, B_r)}{1+r}.$$

Since  $G-F$  vanishes on  $[-\delta, \delta]$  it follows from (4.5) that the last two limits in (4.8) are both zero, and hence

$$(4.9) \quad \lim_{r \rightarrow 1} (G, C_r) = L.$$

Write  $G \sim a_0 + \sum_{k=1}^{\infty} a_k \cos kt$ , and define a function  $h$  on the interval  $[0, 1)$  by

$$h(r) = \sum_{k=0}^{\infty} a_k r^{k+1} = r(G, A_r), \quad 0 \leq r < 1.$$

Since  $|h(r)| \leq \|G\|_{\infty} \|A_r\|_1 = \|G\|_{\infty}$  for  $0 \leq r < 1$ , we see that  $h$  is bounded on  $[0, 1)$ . Now

$$(G, C_r) = \frac{1}{2}(1-r) \left( a_0 + \sum_{k=0}^{\infty} (k+1)a_k r^k \right) = \frac{1}{2}(1-r)(a_0 + h'(r)),$$

and hence by (4.9) we have

$$(4.10) \quad L = \frac{1}{2} \lim_{r \rightarrow 1} (1-r)h'(r).$$

It follows from (4.10) that there exists a number  $\varepsilon > 0$  such that  $h'(r) > L(1-r)^{-1}$  for  $r \geq 1-\varepsilon$ , and hence  $h$  is increasing on the interval  $(1-\varepsilon, 1)$ . By the mean value theorem we know that for any integer  $k \geq 0$  there is a number  $t_k$  in the interval  $[1-2^{-k}\varepsilon, 1-2^{-k-1}\varepsilon]$  such that

$$\begin{aligned} h(1-2^{-k-1}\varepsilon) - h(1-2^{-k}\varepsilon) &= 2^{-k-1}\varepsilon h'(t_k) \\ &> 2^{-k-1}\varepsilon L(1-t_k)^{-1} \geq \frac{1}{2}L. \end{aligned}$$

Thus

$$h(1-2^{-k-1}\varepsilon) - h(1-\varepsilon) = \sum_{j=0}^k (h(1-2^{-j-1}\varepsilon) - h(1-2^{-j}\varepsilon)) \geq \frac{1}{2}(k+1)L.$$

Since  $L \neq 0$  this contradicts the boundedness of  $h$ , and the lemma follows.

**LEMMA 4.11.** *Let  $\{F_r\}$  ( $0 < r < 1$ ) be a family of functions such that  $F_r \rightarrow 0$  uniformly on  $[\delta, 2\pi - \delta]$  for every  $\delta > 0$ , and such that the set  $\{\|F_r \sin t\|_{\infty} : 0 < r < 1\}$  is bounded. Let  $G \in L^1$ , and suppose that there is a neighborhood  $N$  of 0 such that on  $N$  we can write  $G = FH$  where  $F \in L^1$ ,  $H$  is analytic on  $N$  and  $H(0) = 0$ . Then  $\lim_{r \rightarrow 1} (G, F_r) = 0$ .*

**Proof.** Our hypotheses imply that we can write  $G = (1 - e^{it})K$  where  $K \in L^1$ . Hence

$$(4.12) \quad (G, F_r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K(t)(1 - e^{it})F_r(t) dt.$$



It follows from our hypotheses that there exists a constant  $M$  and an  $\varepsilon > 0$  such that  $(1 - e^{it})F_r(t) \leq M$  for all  $t$ , and all  $r > 1 - \varepsilon$ . Thus

$$|K(t)(1 - e^{it})F_r(t)| \leq MK(t)$$

for all  $t$  and all  $r > 1 - \varepsilon$ . The lemma now follows from the dominated convergence theorem.

LEMMA 4.13. Let  $A_r$ ,  $C_r$  be as defined in (4.3), (4.6) and define

$$(4.14) \quad \begin{aligned} D_r(t) &= -(1 - r \cos t)A'_r(t)/(1 + r) \\ &= 2r(1 - r) \sin t(1 - r \cos t)(1 - 2r \cos t + r^2)^{-2}. \end{aligned}$$

Let  $G$  be a function in  $L^1$ , and suppose that there is a neighborhood  $N$  of 0 such that on  $N$  we can write  $G = FH$  where  $F \in L^1$ ,  $H$  is analytic on  $N$ , and  $H(0) = 0$ . Then

$$\lim_{r \rightarrow 1} (G, A_r) = \lim_{r \rightarrow 1} (G, C_r) = \lim_{r \rightarrow 1} (G, D_r) = 0.$$

**Proof.** Let

$$(4.15) \quad G_r(t) = (1 - r) \sin t(1 - 2r \cos t + r^2)^{-1}.$$

Then we can easily verify that

$$(4.16) \quad A_r \sin t = G_r[1 + r],$$

$$(4.17) \quad C_r \sin t = G_r \left[ \frac{(1 - r \cos t)^2 - r^2 \sin^2 t}{(1 - r \cos t)^2 + r^2 \sin^2 t} \right],$$

$$(4.18) \quad D_r \sin t = G_r \left[ \frac{2(1 - r \cos t)r \sin t}{(1 - r \cos t)^2 + r^2 \sin^2 t} \right].$$

The expressions in square brackets on the right-hand sides of (4.16), (4.17) and (4.18) are all bounded by 2 in absolute value, so

$$(4.19) \quad \|A_r \sin t\|_\infty \leq 2\|G_r\|_\infty, \|C_r \sin t\|_\infty \leq 2\|G_r\|_\infty, \|D_r \sin t\|_\infty \leq 2\|G_r\|_\infty.$$

In [10, p. 96] it is shown that there exists a constant  $k$  such that  $A_r(t) \leq k\delta/(\delta^2 + t^2)$  where  $\delta = 1 - r$  and  $|t| \leq \pi$ . Thus

$$|G_r(t)| = (1 + r)^{-1} |\sin t A_r(t)| \leq |t A_r(t)| \leq k\delta t/(\delta^2 + t^2) \leq k$$

for  $|t| \leq \pi$ . Lemma 4.14 thus follows from (4.19) and Lemma 4.11.

LEMMA 4.20. Let  $F$  be a function in  $L^1$  such that  $F_+(0)$  and  $F_-(0)$  both exist, and let  $D_r$  be defined by (4.14). Then

$$\lim_{r \rightarrow 1} \int_0^\pi F(t) D_r(t) dt = F_+(0),$$

$$\lim_{r \rightarrow 1} \int_{-\pi}^0 F(t) D_r(t) dt = -F_-(0),$$

and hence

$$\lim_{r \rightarrow 1} (F, D_r) = \frac{1}{2\pi} (F_+(0) - F_-(0)).$$

**Proof.**  $D_r$  is an odd function that is positive on the interval  $(0, \pi)$ , and  $D_r$  converges uniformly to zero on  $[\delta, 2\pi - \delta]$  for any  $\delta > 0$ . A straightforward calculation shows that

$$\int_0^\pi D_r(t) dt = \frac{2r}{1+r} + (1-r) \log \left( \frac{1+r}{1-r} \right),$$

and hence  $\lim_{r \rightarrow 1} \int_0^\pi D_r(t) dt = 1$ . The lemma follows from these facts by a standard kind of argument.

**LEMMA 4.21.** *Let  $F$  be a function which is continuous at 0, and such that  $F \sin^2 t$  is in  $L^1$ . Then  $\lim_{r \rightarrow 1} (F \sin t, D_r) = 0$ , where  $D_r$  is defined by (4.14).*

**Proof.** Since  $F \sin^2 t$  is in  $L^1$  the inner product  $(F \sin t, D_r)$  exists for  $0 < r < 1$ . Let  $H$  be the periodic function of period  $2\pi$  that agrees with the characteristic function of  $[-\pi/2, \pi/2]$  on  $[-\pi, \pi]$ . Our hypothesis implies that  $FH \sin t$  is in  $L^1$  and  $FH \sin t$  is continuous at 0, and hence by Lemma 4.20

$$(4.22) \quad \lim_{r \rightarrow 1} (FH \sin t, D_r) = 0.$$

Now

$$((1-H)F \sin t, D_r) = 2((1-H)F \sin^2 t, r(1-r)(1-r \cos t)(1-2r \cos t+r^2)^{-2}).$$

Since  $(1-H)F \sin^2 t$  is in  $L^1$ , and  $r(1-r)(1-r \cos t)(1-2r \cos t+r^2)^{-2}$  converges uniformly to zero on the set where  $1-H \neq 0$ , it follows that  $\lim_{r \rightarrow 1} ((1-H)F \sin t, D_r) = 0$ . This fact together with (4.22) proves the lemma.

If  $F$  is a function and  $p$  is a positive integer, define a function  $F^{[p]}$  by

$$(4.23) \quad F^{[p]}(t) = F(pt).$$

Then it is easy to verify that

$$(4.24) \quad (G, F^{[p]}) = \frac{1}{2\pi p} \int_{-\pi}^{\pi} \sum_{j \in J(p)} G((t+2\pi j)/p) \bar{F}(t) dt$$

where

$$(4.25) \quad J(p) = \{n \in \mathbb{Z} : -[\tfrac{1}{2}p] \leq n \leq [\tfrac{1}{2}(p-1)]\}.$$

Here  $[\tfrac{1}{2}p]$  is the greatest integer  $\leq \tfrac{1}{2}p$ .

**LEMMA 4.26.** *Let  $p$  be a positive integer and let  $G$  be an  $L^1$  function such that for every  $j \in J(p)$  either  $G$  is bounded near  $2\pi j/p$ , or else there exists a neighborhood  $N_j$  of  $2\pi j/p$  such that  $G = F_j H_j$  on  $N_j$  where  $F_j \in L^1$ ,  $H_j$  is analytic on  $N_j$ , and  $H_j(2\pi j/p) = 0$ . Suppose that the limit  $\lim_{r \rightarrow 1} (G, C_r^{[p]}) = L$  exists, where  $C_r$  is defined in (4.6). Then  $L = 0$ .*

**Proof.** Write  $J(p)$  as a disjoint union  $J(p) = I \cup K$  where for each  $j \in I$ ,  $G$  is bounded near  $2\pi j/p$ , and for each  $j \in K$ ,  $G$  has a factorization  $G = F_j H_j$  as in the statement of the lemma. Let

$$(4.27) \quad g(t) = \sum_{j \in I} G((t+2\pi j)/p), \quad k(t) = \sum_{j \in K} G((t+2\pi j)/p).$$

Then  $g$  is bounded near 0, and there is a neighborhood  $N$  of 0 such that  $k=fh$  on  $N$  where  $f \in L^1$ ,  $h$  is analytic on  $N$  and  $h(0)=0$ . It follows from (4.24) that

$$(4.28) \quad (G, C_r^{[p]}) = (2\pi p)^{-1} \left( \int_{-\pi}^{\pi} g(t) C_r(t) dt + \int_{-\pi}^{\pi} k(t) C_r(t) dt \right).$$

By lemma 4.13 the second integral in (4.28) goes to zero as  $r \rightarrow 1$ , and hence

$$L = \lim_{r \rightarrow 1} (G, C_r^{[p]}) = (2\pi p)^{-1} \lim_{r \rightarrow 1} \int_{-\pi}^{\pi} g(t) C_r(t) dt.$$

It follows from Lemma 4.7 that  $L=0$ .

**LEMMA 4.29.** *Let  $p$  be a positive integer, and let  $J(p)$  be written as a disjoint union  $J(p)=I \cup K$ . Let  $G$  be a function in  $L^1$  such that  $G$  has left- and right-hand limits at  $2\pi j/p$  for each  $j \in I$ , and for each  $j \in K$  there exists a neighborhood  $N_j$  of  $2\pi j/p$  such that  $G=F_j H_j$  on  $N_j$  where  $F_j \in L^1$ ,  $H_j$  is analytic on  $N_j$  and  $H_j(2\pi j/p)=0$ . Then if  $D_r$  is defined by (4.14) we have*

$$\lim_{r \rightarrow 1} (G, D_r^{[p]}) = (2\pi p)^{-1} \sum_{j \in I} (G_+(2\pi j/p) - G_-(2\pi j/p)).$$

**Proof.** Define functions  $g(t)$  and  $k(t)$  by (4.27). Then

$$(4.30) \quad g_+(0) = \sum_{j \in I} G_+(2\pi j/p), \quad g_-(0) = \sum_{j \in I} G_-(2\pi j/p),$$

and  $k$  has a factorization  $k=fh$  as in the previous lemma. By (4.24) we have

$$(4.31) \quad (G, D_r^{[p]}) = (2\pi p)^{-1} \left( \int_{-\pi}^{\pi} g(t) D_r(t) dt + \int_{-\pi}^{\pi} k(t) D_r(t) dt \right).$$

By Lemma 4.13 the second integral in (4.31) goes to zero as  $r \rightarrow 1$ , and by Lemma 4.20

$$\lim_{r \rightarrow 1} (2\pi p)^{-1} \int_{-\pi}^{\pi} g(t) D_r(t) dt = (2\pi p)^{-1} (g_+(0) - g_-(0)).$$

This result combined with (4.30) and (4.31) proves the lemma.

**5. A formula for  $\lim U_n f(x)$ .** It  $t$  is any real number let  $x(t)$  be the element of  $SU(2)$  defined by

$$(5.1) \quad x(t) = \text{diag} (e^{it}, e^{-it}).$$

If  $f$  is any class function on  $SU(2)$  let  $[f]$  be the function on  $\mathbf{R}$  defined by

$$(5.2) \quad [f](t) = f(x(t)), \quad t \in \mathbf{R}.$$

Then  $[f]$  is an even periodic function of period  $2\pi$ , and if  $F$  is any even periodic function of period  $2\pi$  we can write  $F=[f]$  for some class function  $f$  on  $SU(2)$ . If  $f, g$  are functions on  $SU(2)$  we will write

$$(5.3) \quad (f, g)^* = \int_{SU(2)} f(x) \bar{g}(x) d\mu(x)$$

whenever the integral in (5.3) exists. If  $f$  is a class function on  $SU(2)$  then it follows from [8, pp. 163, 386–389] that  $[f] \sin^2 t \in L^1[-\pi, \pi]$ , and

$$(5.4) \quad \int_{SU(2)} f(x) d\mu(x) = \frac{2}{\pi} \int_0^\pi [f](t) \sin^2 t dt.$$

Hence if  $f$  and  $g$  are class functions on  $SU(2)$  we have

$$(5.5) \quad (f, g)^* = 2([f], [g] \sin^2 t).$$

Let  $Q$  be the projection onto the space of class functions on  $SU(2)$ ,

$$(5.6) \quad Qf(y) = \int_{SU(2)} f(xy x^{-1}) d\mu(x).$$

If  $f \in L^p(SU(2))$  and  $g \in L^q(SU(2))$  where  $p^{-1} + q^{-1} = 1$  then  $(f, Qg)^* = (Qf, g)^*$ . If  $g$  is a class function then

$$(5.7) \quad (f, g)^* = (f, Qg)^* = (Qf, g)^* = 2([Qf], [g] \sin^2 t).$$

LEMMA 5.8. *Let  $U$  be a faithful representation of  $SU(2)$  of type  $(q, p)$ , and let  $f$  be a function in  $L^1(SU(2))$  such that the  $U$ -Fourier series for  $f$  at  $e$  converges to the value  $L$ . Then*

$$L = \lim_{r \rightarrow 1} \sum_{m=q+1}^{p+1} (-1)^{m+p+1} (f, (1-r)Z_{mpr})^*,$$

where

$$(5.9) \quad Z_{mpr} = \sum_{n=0}^{\infty} r^n D_{np+m}.$$

Here  $D_j$  is defined by (3.9).

**Proof.**

$$(5.10) \quad L = \lim_{n \rightarrow \infty} U_n f(e) = \lim_{n \rightarrow \infty} f * D_n^U(e) = \lim_{n \rightarrow \infty} (f, D_n^U)^*.$$

Since the Abel method of finding the limit of a sequence of numbers is regular we have

$$(5.11) \quad L = \lim_{r \rightarrow 1} (1-r) \sum_{n=0}^{\infty} r^n (f, D_{n+1}^U)^* = \lim_{r \rightarrow 1} (f, Y_r)^*,$$

where

$$(5.12) \quad Y_r = (1-r) \sum_{n=0}^{\infty} r^n D_{n+1}^U.$$

For each  $r$  with  $0 < r < 1$  the sum in (5.12) converges absolutely by Corollary 3.20. By formula (3.8) we have

$$(5.13) \quad Y_r = (1-r) \sum_{m=q+1}^{p+1} (-1)^{m+p+1} Z_{mpr} + \epsilon(r)$$

where  $Z_{mpr}$  is defined by (5.9), and  $\varepsilon(r)$  is an error term which arises from the fact that (3.8) may not be valid for  $n < 3$ . Since  $\|\varepsilon(r)\|_\infty \rightarrow 0$  as  $r \rightarrow 1$  we have  $\lim_{r \rightarrow 1} (f, \varepsilon(r))^* = 0$ , and hence the proposition follows from (5.11) and (5.13).

**PROPOSITION 5.14.** *Let  $h$  be a function in  $L^1(SU(2))$  that is continuous at  $e$ . Let  $Z_{mpr}$  be defined by (5.9), and let  $C_r, D_r$  be defined by (4.6) and (4.14). If  $p$  is odd, then*

$$(5.15) \quad \begin{aligned} (h, (1-r)Z_{mpr})^* &= 2p([Qh], \cos \frac{1}{2}t \sin(m + \frac{1}{2})t D_r^{(p)}) \\ &\quad - 2p([Qh], \cos \frac{1}{2}t \cos(m + \frac{1}{2})t C_r^{(p)}) + h(e) + E(r), \end{aligned}$$

where  $\lim_{r \rightarrow 1} E(r) = 0$ . If  $p$  is even then (5.15) holds if we make the additional assumption that  $[Qh] \sin t \in L^1[-\pi, \pi]$ .

**Proof.** Since  $h \in L^1(SU(2))$  it follows that  $Qh \in L^1(SU(2))$  and hence  $[Qh] \sin^2 t \in L^1[-\pi, \pi]$ . Thus, since  $[Qh]$  is continuous at 0, we have  $[Qh]G \in L^1[-\pi, \pi]$  for any analytic function  $G$  of period  $2\pi$  such that  $G(\pi) = G'(\pi) = 0$ . Since  $\cos \pi/2 = \cos(m + \frac{1}{2})\pi = D_r^{(p)}(\pi) = 0$ , the two inner products indicated on the right-hand side of (5.15) exist. It follows from the definition (3.9) of  $D_r$  that

$$Z_{mpr} = \sum_{j=0}^{\infty} r^j \sum_{k=1}^m k \chi_k + \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} r^k \sum_{j=m+1+(n-1)p}^{m+pn} j \chi_j.$$

In this formula write  $\chi_k = \sin k\theta / \sin \theta$  and we get

$$\begin{aligned} (1-r) \sin \theta Z_{mpr} &= \operatorname{Im} \left( \sum_{k=1}^m k e^{ik\theta} + \sum_{n=1}^{\infty} r^n \sum_{j=(m+1)+(n-1)p}^{m+pn} j e^{ij\theta} \right) \\ &= -\frac{d}{d\theta} \left( \operatorname{Re} \left( \frac{(1-r)e^{i(m+1)\theta}}{(e^{i\theta}-1)(1-re^{ip\theta})} + \frac{e^{i\theta}}{1-e^{i\theta}} \right) \right). \end{aligned}$$

Since  $(d/d\theta)(\operatorname{Re}(e^{i\theta}/(1-e^{i\theta}))) = 0$ , this becomes

$$(5.16) \quad \sin \theta Z_{mpr} = -\frac{d}{d\theta} \left[ \frac{1}{2 \sin \theta} \operatorname{Im} \frac{e^{im\theta}(1+e^{i\theta})}{(1-re^{ip\theta})} \right].$$

Now define functions  $H_{mpr}$  on  $SU(2)$  by

$$(5.17) \quad \operatorname{Im} \frac{e^{im\theta}(1+e^{i\theta})}{(1-re^{ip\theta})} = \frac{1}{1-r^2} A_r(p\theta) H_{mpr},$$

where  $A_r$  is defined in (4.3). Then it follows from (5.16) that

$$(5.18) \quad -2(1-r^2) \sin^2 \theta Z_{mpr} = p H_{mpr} A'_r(p\theta) + A_r(p\theta) \sin \theta (d/d\theta) (\csc \theta H_{mpr}).$$

Using the definition (5.17) we can verify that

$$(5.19) \quad H_{mpr} = 2 \cos \frac{1}{2}\theta [r \sin p\theta \cos(m + \frac{1}{2})\theta + (1-r \cos p\theta) \sin(m + \frac{1}{2})\theta].$$

By (5.19), (4.4) and (4.14) we have

$$(5.20) \quad H_{mpr} A'_r(p\theta) = -2 \cos \frac{1}{2}\theta [r^2 \cos(m + \frac{1}{2})\theta B_r(p\theta) + (1+r) \sin(m + \frac{1}{2})\theta D_r(p\theta)].$$

By applying a few trigonometric identities to (5.19) we obtain

$$(5.21) \quad H_{mpr} = \sin \theta \left[ 2 \sin \frac{p\theta}{2} \frac{\cos \frac{1}{2}(p-1-2m)\theta}{\sin \frac{1}{2}\theta} - (1-r) \frac{\sin (p-m-\frac{1}{2})\theta}{\sin \frac{1}{2}\theta} \right],$$

and hence

$$(5.22) \quad \sin \theta (d/d\theta) (\csc \theta H_{mpr}) = 2(1-r)M_{mp} + 2pN_{mp} + 2 \sin \frac{1}{2}p\theta R_{mp},$$

where

$$(5.23) \quad M_{mp} = -\frac{1}{2} \sin \theta (d/d\theta) (\sin (p-m-\frac{1}{2})\theta \csc \frac{1}{2}\theta),$$

$$(5.24) \quad N_{mp} = \cos \frac{1}{2}\theta \cos \frac{1}{2}p\theta \cos \frac{1}{2}(p-1-2m)\theta,$$

$$(5.25) \quad R_{mp} = \sin \theta (d/d\theta) (\cos \frac{1}{2}(p-1-2m)\theta \csc \frac{1}{2}\theta).$$

From (5.18), (5.20), (5.22) and (4.6) we get

$$(5.26) \quad (1-r)Z_{mpr} = \sum_{j=1}^5 \csc^2 \theta F_{rmpj},$$

where

$$(5.27) \quad F_{rmp1} = (r-1)(r+1)^{-1}M_{mp}A_r(p\theta),$$

$$(5.28) \quad F_{rmp2} = -(r+1)^{-1} \sin \frac{1}{2}p\theta R_{mp}A_r(p\theta),$$

$$(5.29) \quad F_{rmp3} = -p(r+1)^{-1}(N_{mp} - \cos \frac{1}{2}\theta \cos (m+\frac{1}{2})\theta)A_r(p\theta),$$

$$(5.30) \quad F_{rmp4} = p \cos \frac{1}{2}\theta \sin (m+\frac{1}{2})\theta D_r(p\theta),$$

$$(5.31) \quad F_{rmp5} = -p \cos \frac{1}{2}\theta \cos (m+\frac{1}{2})\theta C_r(p\theta).$$

LEMMA 5.32. For any function  $h$  in  $L^1(SU(2))$ ,

$$\lim_{r \rightarrow 1} (h, \csc^2 \theta F_{rmp1})^* = 0.$$

**Proof.** By (5.7) we have

$$(5.33) \quad (h, \csc^2 \theta F_{rmp1})^* = 2(r-1)(r+1)^{-1}([Qh][M_{mp}], A_r^{[p]}).$$

We know that  $[Qh] \sin^2 t \in L^1[-\pi, \pi]$ , and since  $[M_{mp}](0) = [M_{mp}]'(0) = [M_{mp}](\pi) = [M_{mp}]'(\pi) = 0$  it follows that  $[Qh][M_{mp}] \in L^1[-\pi, \pi]$ . By (4.24) and (4.25) we have

$$(5.34) \quad ([Qh][M_{mp}], A_r^{[p]}) = (M^*, A_r)$$

where  $M^*$  is the even periodic function of period  $2\pi$  defined by

$$M^*(t) = p^{-1} \sum_{j \in J(p)} [Qh]((t+2\pi j)/p)[M_{mp}]((t+2\pi j)/p).$$

Let  $M^* \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nt$  be the Fourier series for  $M^*$ . Then by (4.3) we have  $(M^*, A_r) = \sum_{n=0}^{\infty} a_n r^n$ . The Riemann-Lebesgue lemma tells us that  $\lim_{n \rightarrow \infty} a_n = 0$  and hence

$$(5.35) \quad \lim_{r \rightarrow 1} (1-r)(M^*, A_r) = \lim_{r \rightarrow 1} (1-r) \sum_{n=0}^{\infty} a_n r^n = \lim_{n \rightarrow \infty} a_n = 0.$$

The lemma follows from (5.33), (5.34) and (5.35).

LEMMA 5.36. *Let  $h$  be a function in  $L^1(SU(2))$  which is continuous at the identity, and let  $F_{rmp2}$  be defined by (5.28). Then for any odd  $p$ ,*

$$(5.37) \quad \lim_{r \rightarrow 1} (h, \csc^2 \theta F_{rmp2})^* = h(e).$$

*If  $p$  is even then (5.37) holds if we make the additional assumption that  $[Qh] \sin t \in L^1[-\pi, \pi]$ .*

**Proof.** By (5.7) we have

$$(5.38) \quad (h, \csc^2 \theta F_{rmp2})^* = -2(\sin \frac{1}{2} pt [R_{mp}][Qh], A_r^{(p)})(r+1)^{-1}.$$

We know that  $[Qh] \sin^2 t \in L^1[-\pi, \pi]$ , and since  $[Qh]$  is continuous at 0 and  $\sin \frac{1}{2} pt [R_{mp}]$  is an everywhere analytic function that vanishes at  $\pi$  together with its derivative, it follows that  $\sin \frac{1}{2} pt [R_{mp}][Qh] \in L^1[-\pi, \pi]$ . By (4.24) and (4.25)

$$(5.39) \quad (\sin \frac{1}{2} pt [R_{mp}][Qh], A_r^{(p)}) = p^{-1} \sum_{j \in J(p)} (-1)^j \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin \frac{1}{2} t f_{pmj}(t) A_r(t) dt,$$

where

$$f_{pmj}(t) = [Qh]((t+2\pi j)/p)[R_{mp}]((t+2\pi j)/p).$$

Let

$$(5.40) \quad I(r, j, p, m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin \frac{1}{2} t f_{pmj}(t) A_r(t) dt.$$

If the interval  $[(2j-1)\pi/p, (2j+1)\pi/p]$  does not contain 0 or  $\pm\pi$  then  $f_{pmj} \in L^1[-\pi, \pi]$  and hence by Lemma 4.13

$$(5.41) \quad \lim_{r \rightarrow 1} I(r, j, p, m) = 0, \quad j \in J(p), j \neq 0, -\frac{1}{2}p, \pm \frac{1}{2}(p-1).$$

Now

$$(5.42) \quad I(r, 0, p, m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin \frac{1}{2} t [Qh](t/p)[R_{mp}](t/p) A_r(t) dt,$$

and since  $\sin \frac{1}{2} t [Qh](t/p)[R_{mp}](t/p)$  is continuous at 0 if we define its value at 0 to be  $-p[Qh](0) = -ph(e)$ , it follows from (5.42) and (4.5) that

$$(5.43) \quad \lim_{r \rightarrow 1} I(r, 0, p, m) = -ph(e).$$

Now suppose  $p$  is odd. Then  $\pm \frac{1}{2}(p-1) \in J(p)$  but  $-\frac{1}{2}p \notin J(p)$ . Since  $\sin \frac{1}{2} t f_{pmj} \in L^1[-\pi, \pi]$  for all  $j$ , and

$$f_{pm, \pm(p-1)/2} = [Qh]((t \pm \pi(p-1))/p)[R_{mp}]((t \pm \pi(p-1))/p)$$

is locally in  $L^1$  near  $t=0$ , it follows from Lemma 4.13 that

$$(5.44) \quad \lim_{r \rightarrow 1} I(r, \pm \frac{1}{2}(p-1), p, m) = 0.$$

Equation (5.37) for odd  $p$  follows from (5.38)–(5.41) and (5.43) and (5.44). Now suppose  $p$  is even, so that  $-\frac{1}{2}p \in J(p)$  but  $\pm\frac{1}{2}(p-1) \notin J(p)$ . Then

$$I(r, -\frac{1}{2}p, p, m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin \frac{1}{2}t [Qh] \left( \frac{t}{p} - \pi \right) [R_{mp}] \left( \frac{t}{p} - \pi \right) A_r(t) dt.$$

The assumption that  $[Qh] \sin t \in L^1[-\pi, \pi]$  implies that  $[Qh](t/p - \pi)[R_{mp}(t/p - \pi)] \in L^1[-\pi, \pi]$ , and hence it follows from Lemma 4.13 that  $\lim_{r \rightarrow 1} I(r, -\frac{1}{2}p, p, m) = 0$ . This result combined with (5.38)–(5.41) and (5.43) proves (5.37) for even  $p$ .

**LEMMA 5.45.** *Let  $h$  be a function in  $L^1(SU(2))$  which is continuous at the identity, and let  $F_{rmp3}$  be defined by (5.29). Then for any odd  $p$ ,  $\lim_{r \rightarrow 1} (h, \csc^2 \theta F_{rmp3})^* = 0$ . If  $p$  is even the same result holds if we make the additional assumption that  $[Qh] \sin t \in L^1[-\pi, \pi]$ .*

**Proof.** The proof is almost identical with the proof of the previous lemma, so we omit it.

*Completion of the proof of Proposition 5.14.* It follows from formula (5.7) and equations (5.30), (5.31) that

$$(5.46) \quad (h, \csc^2 \theta F_{rmp4})^* = 2p([Qh] \cos \frac{1}{2}t \sin(m + \frac{1}{2})t, D_r^{[p]})$$

$$(5.47) \quad (h, \csc^2 \theta F_{rmp5})^* = -2p([Qh] \cos \frac{1}{2}t \cos(m + \frac{1}{2})t, C_r^{[p]}).$$

The inner products indicated in (5.46) exist for any  $h$  in  $L^1(SU(2))$  because  $D_r^{[p]}(0) = D_r^{[p]}(\pi) = \sin(m + \frac{1}{2})(0) = \cos \pi/2 = 0$ , and the inner products indicated in (5.47) exist for any  $h \in L^1(SU(2))$  which is continuous at 0 because  $\cos \pi/2 = \cos(m + \frac{1}{2})\pi = 0$ . Let  $h$  be a function in  $L^1(SU(2))$  that is continuous at  $e$ . Then by (5.26)

$$(h, (1-r)Z_{mpr})^* = \sum_{j=1}^5 (h, \csc^2 \theta F_{rmpj})^*.$$

Proposition 5.14 follows from this formula and Lemmas 5.32, 5.36, 5.45 and equations (5.46) and (5.47).

**LEMMA 5.48.** *Let  $U$  be a faithful representation of  $SU(2)$  of type  $(q, p)$ , and let  $f$  be a function in  $L^1(SU(2))$  such that  $f$  is continuous at  $e$ , and the  $U$ -Fourier series for  $f$  at  $e$  converges to the value  $L$ . If  $p$  is odd then*

$$(5.49) \quad \begin{aligned} L = 2p \lim_{r \rightarrow 1} [ & (\sin \frac{1}{2}(p+q+3)t \cos \frac{1}{2}(p-q+1)t [Qf], D_r^{[p]}) \\ & - (\cos \frac{1}{2}(p+q+3)t \cos \frac{1}{2}(p-q+1)t [Qf], C_r^{[p]}) ] + f(e). \end{aligned}$$

*If  $p$  is even then (5.49) holds if we make the additional assumption that  $[Qf] \sin t$  is in  $L^1[-\pi, \pi]$ .*



**Proof.** The result follows from Lemma 5.8, Proposition 5.14 and the trigonometric identities

$$(5.50) \quad \begin{aligned} \cos \frac{1}{2}t \sum_{m=q+1}^{p+1} (-1)^{m+p+1} \sin(m+\frac{1}{2})t &= \sin \frac{1}{2}(p+q+3)t \cos \frac{1}{2}(p-q+1)t; \\ \cos \frac{1}{2}t \sum_{m=q+1}^{p+1} (-1)^{m+p+1} \cos(m+\frac{1}{2})t &= \cos \frac{1}{2}(p+q+3)t \cos \frac{1}{2}(p-q+1)t. \end{aligned}$$

**LEMMA 5.51.** *Let  $F$  be an even analytic function of period  $2\pi$  such that  $F(\pi) = F'(\pi) = 0$ , let  $f$  be a function in  $L^1(SU(2))$  that is continuous at the identity, and let  $p$  be an integer. Suppose that for each integer  $j$  satisfying  $1 \leq j \leq [\frac{1}{2}(p-1)]$  either  $[Qf]$  is bounded near  $(2\pi j/p)$  or  $F(2\pi j/p) = 0$ . Suppose also that  $\lim_{r \rightarrow 1} (F[Qf], C^{[p]})$  exists. If  $p$  is odd then*

$$(5.52) \quad \lim_{r \rightarrow 1} (F[Qf], C^{[p]}) = 0.$$

*If  $p$  is even then (5.52) holds if we make the additional assumption that  $[Qf] \sin t$  is in  $L^1[-\pi, \pi]$ .*

**Proof.** The assumptions that  $F(\pi) = F'(\pi) = 0$  and that  $f$  is continuous at  $e$  imply that  $F[Qf]$  is in  $L^1[-\pi, \pi]$ . Let  $\bar{J}(p) = \{j \in \mathbf{Z} : |j| \leq [\frac{1}{2}(p-1)]\}$ . Since  $F[Qf]$  is an even function of period  $2\pi$ , and  $[Qf]$  is locally in  $L^1$  at each point of the open interval  $(-\pi, \pi)$ , our assumptions imply that for each integer  $j$  in  $\bar{J}(p)$  either  $F[Qf]$  is bounded near  $2\pi j/p$  or else there exists a neighborhood  $N_j$  of  $2\pi j/p$  such that  $[Qf]$  is in  $L^1$  on  $N_j$  and  $F(2\pi j/p) = 0$ . If  $p$  is odd then  $\bar{J}(p) = J(p)$  where  $J(p)$  is defined in (4.25), and hence (5.52) holds for odd  $p$  by Lemma 4.26. If  $p$  is even then  $J(p) = \bar{J}(p) \cup \{-[\frac{1}{2}p]\}$ . If we assume that  $[Qf] \sin t$  is in  $L^1[-\pi, \pi]$  then since  $F(\pi) = F'(\pi) = 0$  it follows that  $F[Qf]$  can be written as the product of a function which is analytic at  $-\pi = 2\pi(-[\frac{1}{2}p]/p)$  and vanishes at  $-\pi$ , and a function in  $L^1[-\pi, \pi]$ . Hence in this case (5.52) again follows from Lemma 4.26.

**LEMMA 5.53.** *Let  $F$  and  $H$  be respectively even and odd analytic functions of period  $2\pi$ , and suppose that  $F(\pi) = F'(\pi) = 0$ . Let  $f$  be a function in  $L^1(SU(2))$  that is continuous at the identity, let  $p$  be an integer and let  $J^*(p) = \{n \in \mathbf{Z} : 1 \leq n \leq [\frac{1}{2}(p-1)]\}$ . Suppose that  $J^*(p)$  can be written as a disjoint union  $J^*(p) = I^* \cup K^* \cup L^*$  where for each  $j$  in  $I^*$ ,  $[Qf]$  has left- and right-hand limits at  $2\pi j/p$ ; for each  $j$  in  $K^*$ ,  $F(2\pi j/p) = H(2\pi j/p) = 0$ ; and for each  $j$  in  $L^*$ ,  $H(2\pi j/p) = 0$  and  $[Qf]$  is bounded near  $2\pi j/p$ . Suppose that the limit*

$$(5.54) \quad \lim_{r \rightarrow 1} (2p(H[Qf], D_r^{[p]}) - 2p(F[Qf], C_r^{[p]})) = L$$

*exists. If  $p$  is odd then*

$$(5.55) \quad L = \frac{2}{\pi} \sum_{j \in I^*} H(2\pi j/p) ([Qf]_+(2\pi j/p) - [Qf]_-(2\pi j/p)).$$

If  $p$  is even then (5.55) still holds if we make the additional assumption that

$$[Qf] \in L^1[-\pi, \pi].$$

**Proof.** The inner product  $(H[Qf], D_r^{(p)})$  exists because  $H(\pi) = D_r^{(p)}(\pi) = 0$  (we use the fact that any odd function of period  $2\pi$  vanishes at  $\pi$ ), and we saw in the previous lemma that the other inner product in (5.54) exists. Let  $N$  be the interval  $[-\pi + \frac{1}{2}\pi p^{-1}, \pi - \frac{1}{2}\pi p^{-1}]$ , and let  $M$  be the complement of  $N$  in  $[-\pi, \pi]$ . Let  $K_M, K_N$  be the characteristic functions of  $M, N$  respectively, made periodic of period  $2\pi$ . Let  $P = H[Qf]K_M$ ,  $G = H[Qf]K_N$  where  $H$  is as in the statement of the lemma. Then  $G \in L^1[-\pi, \pi]$  and  $G$  vanishes on a neighborhood of  $-\pi$ . Since  $H(\pi) = 0$  it follows that  $P \sin t \in L^1[-\pi, \pi]$ . Apply Lemma 4.29 to the function  $G$  with  $I = \{0\} \cup I^* \cup -I^*$  and  $K = K^* \cup -K^* \cup L^* \cup -L^* \cup (\{-\frac{1}{2}p\} \cap J(p))$ , and we obtain

$$\begin{aligned} \lim_{r \rightarrow 1} 2p(G, D_r^{(p)}) &= \frac{1}{\pi} \sum_{j \in I} (G_+(2\pi j/p) - G_-(2\pi j/p)) \\ (5.56) \qquad \qquad \qquad &= \frac{2}{\pi} \sum_{j \in I^*} H(2\pi j/p) ([Qf]_+(2\pi j/p) - [Qf]_-(2\pi j/p)). \end{aligned}$$

Now

$$(5.57) \qquad 2p(P, D_r^{(p)}) = \frac{1}{\pi} \int_{-\pi}^{\pi} P\left(\frac{t}{p} - \pi\right) D_r(t) dt \qquad \text{for } p \text{ even,}$$

$$(5.58) \qquad 2p(P, D_r^{(p)}) = \frac{\varepsilon(p)}{\pi} \int_{-\pi}^{\pi} P\left(\frac{t+\pi}{p} - \pi\right) D_r(t) dt \qquad \text{for } p \text{ odd,}$$

where  $\varepsilon(p) = 1$  if  $p = 1$  and  $\varepsilon(p) = 2$  if  $p \geq 3$ . (To derive (5.58) we have used the fact that  $P$  and  $D_r$  are both odd functions.) It follows from the facts that  $P$  vanishes on  $[-\pi + \frac{1}{2}\pi p^{-1}, \pi - \frac{1}{2}\pi p^{-1}]$  and  $P \sin t \in L^1[-\pi, \pi]$  that  $P((t+\pi)/p - \pi)$  vanishes on a neighborhood of  $t = 0$ , and  $P((t+\pi)/p - \pi) \sin t \in L^1[-\pi, \pi]$ . Hence we can apply Lemma 4.21 to the function  $P((t+\pi)/p - \pi) \csc t$  to show that

$$(5.59) \qquad \lim_{r \rightarrow 1} 2p(P, D_r^{(p)}) = 0 \qquad \text{for } p \text{ odd.}$$

Suppose now that  $p$  is even and  $[Qf]$  is in  $L^1[-\pi, \pi]$ . Then we can write  $P(t/p - \pi)$  as the product of a function  $[Qf](t/p - \pi)K_M(t/p - \pi)$  in  $L^1[-\pi, \pi]$  and an analytic function  $H(t/p - \pi)$  that vanishes at 0. Hence by Lemma 4.13

$$(5.60) \qquad \lim_{r \rightarrow 1} 2p(P, D_r^{(p)}) = 0 \qquad \text{for } p \text{ even if } [Qf] \in L^1.$$

Since  $P + G = H[Qf]$ , it follows from (5.56)–(5.60) that

$$(5.61) \qquad \lim_{r \rightarrow 1} 2p(H[Qf], D_r^{(p)}) = \frac{2}{\pi} \sum_{j \in I^*} H(2\pi j/p) ([Qf]_+(2\pi j/p) - [Qf]_-(2\pi j/p)).$$

This result combined with our assumption (5.54) shows that  $\lim_{r \rightarrow 1} (F[Qf], C_r^{[p]})$  exists. By Lemma 5.51 we thus have

$$(5.62) \quad \lim_{r \rightarrow 1} (F[Qf], C_r^{[p]}) = 0,$$

and Lemma 5.53 follows from (5.61) and (5.62).

**THEOREM 5.63.** *Let  $U$  be a faithful representation of  $SU(2)$  of type  $(q, p)$ , let  $x \in SU(2)$ , and let  $f$  be a function in  $L^1(SU(2))$  such that  $f$  is continuous at  $x$  and the  $U$ -Fourier series for  $f$  at  $x$  converges to the limit  $L$ . Let  ${}_x f$  be the function on  $SU(2)$  defined by  ${}_x f(y) = f(xy)$  and let*

$$(5.64) \quad J(q, p) = \left\{ j \in \mathbb{Z} : 1 \leq j \leq [\tfrac{1}{2}(p-1)], \sin\left(\frac{j\pi(q+3)}{p}\right) \cos\left(\frac{j\pi(q-1)}{p}\right) \neq 0 \right\}.$$

*Suppose that  $[Q({}_x f)]$  has left- and right-hand limits at  $2\pi j/p$  for each  $j \in J(q, p)$ , and  $[Q({}_x f)]$  is bounded at  $2\pi j/p$  for all  $j$  such that  $1 \leq j \leq [\tfrac{1}{2}(p-1)]$  and  $\sin(j\pi(q+3)/p) = 0$  but  $\cos(j\pi(q-1)/p) \neq 0$ . Then if  $p$  is odd*

$$L = f(x) + \frac{2}{\pi} \sum_{j \in J(q, p)} \sin\left(\frac{(q+3)j\pi}{p}\right) \cos\left(\frac{(q-1)j\pi}{p}\right) \left( [Q({}_x f)]_+ \left(\frac{2\pi j}{p}\right) - [Q({}_x f)]_- \left(\frac{2\pi j}{p}\right) \right),$$

*and if  $p$  is even the same formula holds provided we make the additional assumption that  $[Q({}_x f)]$  is in  $L^1[-\pi, \pi]$ .*

**Proof.** Since  $f$  is continuous at  $x$ , we see that  ${}_x f$  is continuous at  $e$ , and since  $U_n f(x) = U_n({}_x f)(e)$  it follows that the  $U$ -Fourier series for  ${}_x f$  at  $e$  converges to  $L$ . The theorem follows from Lemma 5.48 and Lemma 5.53 with  $I^* = J(p, q)$  and  $K^* = \{j \in J^*(p) : \cos j\pi(q-1)/p = 0\}$ .

**COROLLARY 5.65.** *Let  $U$  be any faithful representation of  $SU(2)$ , and let  $C(SU(2))$  be the space of continuous functions on  $SU(2)$ . Then  $U$ -Fourier series are honest for functions in  $C(SU(2))$ .*

**Proof.** This follows immediately from Theorem 5.63 since  $Q({}_x f)$  is continuous for any  $x \in SU(2)$  and any  $f \in C(SU(2))$ .

**COROLLARY 5.66.** *Let  $U$  be a faithful representation of  $SU(2)$ . Then  $U$ -Fourier series are honest for functions in  $L^\infty(SU(2))$  if and only if  $U$  is of type  $(1, 1)$ ,  $(2, 2)$  or  $(3, 3)$ .*

**Proof.** We verify immediately from (5.64) that  $J(1, 1) = J(2, 2) = J(3, 3) = \emptyset$ , so it follows from Theorem 5.63 that  $U$ -Fourier series are honest for functions in  $L^\infty(SU(2))$  if  $U$  is of type  $(1, 1)$ ,  $(2, 2)$  or  $(3, 3)$ . On the other hand let  $U$  be of type

$(q, p)$  different from  $(1, 1)$ ,  $(2, 2)$  or  $(3, 3)$  and for  $1 \leq j \leq \frac{1}{2}[p-1]$  let  $f_{jp}$  be the bounded class function on  $SU(2)$  defined by

$$\begin{aligned}
 f_{jp}(x) &= \theta(x) \csc \theta(x) && \text{if } 0 < \theta(x) \leq 2\pi j/p \\
 &= (\theta(x) - \pi) \csc \theta(x) && \text{if } 2\pi j/p < \theta(x) < \pi \\
 (5.67) \quad &= 1 && \text{if } x = e \\
 &= -1 && \text{if } x = -e.
 \end{aligned}$$

Then  $f_{jp}$  is continuous at  $e$ . Using (3.6) and (5.4) we see that

$$\begin{aligned}
 (5.68) \quad \left( f_{jp} * \sum_{k=1}^n k \chi_k \right)(e) &= \sum_{k=1}^n k (f_{jp}, \chi_k)^* = -2 \sum_{k=1}^n \cos(2\pi jk/p) \\
 &= 1 - \csc(\pi j/p) \sin((n + \frac{1}{2})(2\pi j/p)).
 \end{aligned}$$

Using this result in (3.8), and noting the trigonometric identity (5.50) we get

$$\begin{aligned}
 (5.69) \quad U_n(f_{jp})(e) &= \sum_{m=q+1}^{p+1} (-1)^{m+p+1} (f_{jp} * D_{(n-1)p+m})(e) \\
 &= 1 - \csc(\pi j/p) \sum_{m=q+1}^{p+1} (-1)^{m+p+1} \sin((m + \frac{1}{2})(2\pi j/p)) \\
 &= 1 - 2 \csc(2\pi j/p) \sin((q+3)\pi j/p) \cos((q-1)\pi j/p)
 \end{aligned}$$

for all  $n \geq 3$ . Note that the right-hand side of (5.69) does not depend on  $n$ . It follows from (5.69) and our restrictions on  $(q, p)$  that the  $U$ -Fourier series for  $f_{1p}$  converges deceptively at  $e$  unless  $p=2(q-1)$ . If  $p=2(q-1)$ , then our restrictions on  $(q, p)$  imply that  $p \geq 6$  so  $f_{2p}$  is defined, and (5.69) shows that the  $U$ -Fourier series for  $f_{2p}$  converges deceptively at  $e$ . Corollary 5.66 follows from these remarks.

**PROPOSITION 5.70.** *Let  $a, b$  be integers satisfying  $a > b \geq 0$ . Let  $f$  be a bounded function in  $L^1(SU(2))$  which is continuous at  $x \in SU(2)$ , and suppose that the limit*

$$L = \lim_{n \rightarrow \infty} \sum_{k=1}^{an+b} (f * k \chi_k)(x)$$

*exists. If  $a \leq 2$  or if  $a=2b+1$  we can conclude that  $L=f(x)$ . For any other pair  $(a, b)$  with  $a > b \geq 0$  there is a bounded function  $f$  such that  $L \neq f(x)$ .*

**Proof.**  $L = \lim_{n \rightarrow \infty} ({}_x f, D_{an+b})^* = \lim_{r \rightarrow 1} ({}_x f, (1-r)Z_{bar})^*$  where  $Z_{bar}$  is defined by (5.9). Suppose  $a=2b+1$ . Then  $\sin(b + \frac{1}{2})(2\pi j/a) = 0$  for all integers  $j$ . By Proposition 5.14 and Lemma 5.53 (with  $p=a$ ,  $H = \cos \frac{1}{2}t \sin(b + \frac{1}{2})t$ ,  $F = \cos \frac{1}{2}t \cos(b + \frac{1}{2})t$ , and  $K^* = I^* = \emptyset$ ) we get  $L = {}_x f(e) = f(x)$ . If  $a \leq 2$  we again use Proposition 5.14 and Lemma 5.53 (but now  $J^*(p) = \emptyset$ ) to get  $L = f(x)$ . Now suppose  $a > 2$  and  $a \neq 2b+1$ .

Then  $\sin(b + \frac{1}{2})(2\pi/a) \neq 0$ . Let  $f_{1a}$  be defined by (5.67). Then for any integer  $n$  we have by (5.68)

$$\sum_{k=1}^{an+b} (f_{1a} * k\chi_k)(e) = 1 - \csc(\pi/a) \sin(b + \frac{1}{2})(2\pi/a)$$

and hence  $L \neq f(e)$ .

**6. Honesty of  $U$ -Fourier series for a certain class of functions.** Let  $A, B$  be the functions on  $SU(2)$  defined by

$$(6.1) \quad x = \begin{bmatrix} A(x) & B(x) \\ -\bar{B}(x) & \bar{A}(x) \end{bmatrix}, \quad x \in SU(2).$$

If  $x$  and  $y$  are elements of  $SU(2)$  define

$$(6.2) \quad d(x, y) = [\text{Tr}(x-y)(x-y)^*]^{1/2}.$$

Then  $d$  is a metric on  $SU(2)$  such that the map of  $SU(2)$  into  $C^4$  defined by  $x \rightarrow (A(x), B(x), \bar{A}(x), \bar{B}(x))$  is an isometry for the Euclidean metric on  $C^4$ . This metric is easily seen to be left and right translation invariant. We will denote the open ball of radius  $r$  about  $x \in SU(2)$  for this metric by  $B(x, r)$ . In the following discussion the Hausdorff dimension of subsets of  $SU(2)$  will be with respect to the metric  $d$ . (See Chapter VII of [3] for definition and properties of Hausdorff dimension and Hausdorff measure.)

**PROPOSITION 6.3.** *Let  $S$  be the set of all functions in  $L^\infty(SU(2))$  whose set of discontinuities has Hausdorff dimension  $\leq 2$ , and let  $T$  be the set consisting of those functions in  $S$  whose set of discontinuities has Hausdorff dimension  $< 2$ . Let  $U$  be a faithful representation of  $SU(2)$ . Then  $U$ -Fourier series are honest for functions in  $T$ , but  $U$ -Fourier series are honest for functions in  $S$  if and only if  $U$  is of type  $(1, 1)$ ,  $(2, 2)$  or  $(3, 3)$ .*

**Proof.** Let  $U$  be of type  $(q, p)$ , let  $f \in T$ , and let  $x$  be a point of continuity of  $f$ . We will show that  $[Q(xf)]$  is continuous on  $[0, \pi)$ , and in particular  $[Q(xf)]$  is continuous on  $(2\pi/p)J(q, p)$  (see (5.64)). It will then follow from Theorem 5.63 that if the  $U$ -Fourier series for  $f$  at  $x$  converges, it must converge to  $f(x)$ . Note that  $[Q(xf)]$  is continuous at 0 because  $xf$  is continuous at  $e$ . Let  $E$  be the set of points at which  $xf$  is discontinuous. Since  $T$  is translation invariant the Hausdorff dimension of  $E$  is less than 2, and hence the two-dimensional measure of  $E$  is zero. This means that for each positive integer  $n$  there is a sequence  $\{x_{nk} : 1 \leq k < \infty\}$  of points in  $E$  and a sequence  $\{r_{nk} : 1 \leq k < \infty\}$  of positive numbers such that

$$(6.4) \quad E \subseteq \bigcup_{k=1}^{\infty} B(x_{nk}, r_{nk}) \quad \text{and} \quad \sum_{k=1}^{\infty} r_{nk}^2 < \frac{1}{n}.$$

Let  $B_n = \bigcup_{k=1}^{\infty} B(x_{nk}, r_{nk})$ . Then the complement  $B'_n$  of  $B_n$  is a closed set on which  $xf$  is continuous, and by the Tietze extension theorem we can find a continuous

function  $f_n$  on  $SU(2)$  such that  $\|f_n\|_\infty \leq \|_xf\|_\infty = \|f\|_\infty$  and  $_xf = f_n$  on  $B'_n$ . Since  $f_n$  is continuous,  $[Q(f_n)]$  is also continuous. For any  $t \in [0, \pi]$

$$(6.5) \quad [Qf_n](t) - [Q(_xf)](t) = \int_{SU(2)} f_n(yx(t)y^{-1}) - _xf(yx(t)y^{-1}) d\mu(y)$$

(see (5.2) and (5.6)). Since  $f_n - _xf$  vanishes off of  $B_n$  we get

$$(6.6) \quad |[Qf_n](t) - [Q(_xf)](t)| \leq 2\|f\|_\infty \mu(E_n)$$

where

$$(6.7) \quad E_n = \{y : yx(t)y^{-1} \in B_n\} \subseteq \bigcup_{k=1}^{\infty} \{y : yx(t)y^{-1} \in B(x_{nk}, r_{nk})\}.$$

It follows from (6.7) that

$$(6.8) \quad \mu(E_n) \leq \sum_{k=1}^{\infty} \mu\{y : yx(t)y^{-1} \in B(x_{nk}, r_{nk})\}.$$

In order to estimate the sum (6.8) we will need to prove a few lemmas.

**LEMMA 6.9.** *Let  $A$  and  $B$  be the functions on  $SU(2)$  defined in (6.1). Then there exists a constant  $K$  such that  $\mu\{x \in SU(2) : |B(x)| < \varepsilon\} < K\varepsilon^2$  for every  $\varepsilon > 0$ .*

**Proof.** If  $x = (x_1, x_2, x_3, x_4)$  is any point in  $\mathbf{R}^4$ , let  $\bar{x}$  be the matrix defined by

$$(6.10) \quad \bar{x} = \begin{bmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{bmatrix}.$$

Then  $SU(2)$  acts as a group of orthogonal transformations on  $\mathbf{R}^4$  by left translations,  $\bar{x} \rightarrow g\bar{x}$  for  $g \in SU(2)$ . Also we may identify the unit sphere  $S^3$  in  $\mathbf{R}^4$  with  $SU(2)$  by (6.10). Now  $S^3$  can be parametrized by

$$(6.11) \quad \begin{aligned} x_1 &= \sin \theta_1, & x_2 &= \cos \theta_1 \sin \theta_2, & -\frac{\pi}{2} &\leq \theta_1, \theta_2 \leq \frac{\pi}{2} \\ x_3 &= \cos \theta_1 \cos \theta_2 \sin \theta_3, & x_4 &= \cos \theta_1 \cos \theta_2 \cos \theta_3, & 0 &\leq \theta_3 < 2\pi. \end{aligned}$$

In [1, p. 116] it is shown that there is a unique measure  $d\omega$  on  $S^3$  which is invariant under all orthogonal maps of  $\mathbf{R}^4$  and has total mass = 1, and this measure is

$$(6.12) \quad d\omega = \frac{1}{2}\pi^{-2} \cos^2 \theta_1 \cos \theta_2 d\theta_1 d\theta_2 d\theta_3.$$

Thus  $d\omega$  is Haar measure on  $SU(2)$ . Using the identifications (6.10) and (6.11) we have  $|B|^2 = x_3^2 + x_4^2 = \cos^2 \theta_1 \cos^2 \theta_2$ , and hence

$$(6.13) \quad \begin{aligned} \mu\{x : |B(x)| < \varepsilon\} &= \frac{1}{2}\pi^{-2} \int_R \cos^2 \theta_1 \cos \theta_2 d\theta_1 d\theta_2 d\theta_3 \\ &\leq 4\varepsilon\pi^{-1} \int_V \cos \theta_1 d\theta_1 d\theta_2 \end{aligned}$$

where

$$R = \{(\theta_1, \theta_2, \theta_3) : \cos \theta_1 \cos \theta_2 < \varepsilon, -\pi/2 \leq \theta_1, \theta_2 \leq \pi/2, 0 \leq \theta_3 < 2\pi\},$$

$$V = \{(\theta_1, \theta_2) : \cos \theta_1 \cos \theta_2 < \varepsilon, 0 \leq \theta_1, \theta_2 \leq \pi/2\}.$$

Now

$$\int_V \cos \theta_1 d\theta_1 d\theta_2 = \int_0^{\arccos \varepsilon} (\arcsin(\varepsilon \sec \theta_1)) \cos \theta_1 d\theta_1 + \int_{\arccos \varepsilon}^{\pi/2} \cos \theta_1 d\theta_1,$$

and since  $\arcsin t \leq \frac{1}{2}\pi t$  for  $0 \leq t \leq 1$  we see that  $\int_V \cos \theta_1 d\theta_1 d\theta_2 < 3\varepsilon$ . This result combined with (6.13) proves Lemma 6.9.

LEMMA 6.14. Let  $z \in SU(2)$ , let  $t \in (0, \pi)$  and let  $\varepsilon$  be a positive number. Let

$$F(t, z, \varepsilon) = \{y \in SU(2) : yx(t)y^{-1} \in B(z, \varepsilon)\}.$$

Then there exists an absolute constant  $K$  such that

$$\mu(F(t, z, \varepsilon)) \leq K\varepsilon^2 \csc^2 t.$$

**Proof.** Let  $y_0 \in F(t, z, \varepsilon)$  (if  $F(t, z, \varepsilon) = \emptyset$  the lemma is clearly true). If  $y$  is any point in  $F(t, z, \varepsilon)$  then

$$(6.15) \quad \begin{aligned} d(y_0^{-1}yx(t), x(t)y_0^{-1}y) &= d(yx(t)y^{-1}, y_0x(t)y_0^{-1}) \\ &\leq d(yx(t)y^{-1}, z) + d(z, y_0x(t)y_0^{-1}) < 2\varepsilon. \end{aligned}$$

A straightforward calculation shows that for any  $a \in SU(2)$

$$(6.16) \quad d(ax(t), x(t)a) = 8^{1/2}|B(a)| \sin t.$$

If we take  $a = y_0^{-1}y$  in (6.16) and use (6.15) we obtain

$$(6.17) \quad |B(y_0^{-1}y)| < \varepsilon \csc t \cdot 2^{-1/2} < \varepsilon \csc t, \quad y \in F(t, z, \varepsilon).$$

Let  $H(t, \varepsilon) = \{x \in SU(2) : |B(x)| < \varepsilon \csc t\}$ . It follows from (6.17) that  $F(t, z, \varepsilon) \subseteq y_0 H(t, \varepsilon)$ , and hence by Lemma 6.9

$$\mu(F(t, z, \varepsilon)) \leq \mu(H(t, \varepsilon)) \leq K\varepsilon^2 \csc^2 t$$

which proves Lemma 6.14.

Applying Lemma 6.14 to (6.8) we get

$$(6.18) \quad \mu(E_n) \leq K \csc^2 t \sum_{k=1}^{\infty} r_{nk}^2 \leq Kn^{-1} \csc^2 t$$

so by (6.6) we have

$$|[Qf_n](t) - [Q_x f](t)| \leq 2K \|f\|_{\infty} n^{-1} \csc^2 t.$$

It follows that  $[Qf_n]$  converges uniformly to  $[Q_x f]$  on any compact subinterval of  $(0, \pi)$ , and hence the limit function  $Q_x f$  is continuous on  $(0, \pi)$ . This completes the proof that  $U$ -Fourier series are honest for functions in  $T$ . It follows from Corollary

5.66 that  $U$ -Fourier series are honest for functions in  $S$  if  $U$  is of type  $(1, 1)$ ,  $(2, 2)$  or  $(3, 3)$ . If  $U$  is of type  $(q, p)$  different from  $(1, 1)$ ,  $(2, 2)$  or  $(3, 3)$ , then we saw in the proof of Corollary 5.66 that some function  $f_{jp}$  defined as in (5.67) has a  $U$ -Fourier series that converges deceptively at  $e$ , so to complete the proof of Proposition 6.3 it will suffice to show that  $f_{jp} \in S$ . The set of discontinuities of  $f_{jp}$  is

$$\Delta(j, p) = \{x \in SU(2) : \theta(x) = 2\pi j/p\}.$$

Use (6.10) to identify  $SU(2)$  with the unit sphere in  $\mathbb{R}^4$ . Then by (3.5) we see that  $\Delta(j, p)$  is the intersection of the sphere  $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$  and the plane  $x_1 = \cos(2\pi j/p)$ . This intersection is a 2-sphere which has Hausdorff dimension 2 for the Euclidean metric on  $\mathbb{R}^4$ . Since  $d(\bar{x}, \bar{y}) = 2^{1/2}|x - y|$  for all  $x, y \in S^3 \subseteq \mathbb{R}^4$  we conclude that the Hausdorff dimension of  $\Delta(j, p)$  is also 2, so  $f_{jp} \in S$ . This completes the proof of Proposition 6.3.

**7. Deception sets.** Let  $U$  be a faithful representation of  $SU(2)$  of type  $(q, p)$ . The *deception set*  $D_U$  or  $D(q, p)$  of  $U$  is the subset of  $SU(2)$  given by

$$(7.1) \quad \begin{aligned} D(q, p) &= \{x : e^{ip\theta(x)} = 1\} && \text{if } p \text{ is odd} \\ &= \{-e\} \cup \{x : e^{ip\theta(x)} = 1 \text{ and } e^{i(q-1)\theta(x)} \neq -1\} && \text{if } p \text{ is even.} \end{aligned}$$

Since two elements  $x, y$  of  $SU(2)$  are conjugate if and only if  $\theta(x) = \theta(y)$  we see that  $x D_U = D_U x$  for all  $x \in SU(2)$ .

**THEOREM 7.2.** *Let  $U$  be a faithful representation of  $SU(2)$  with deception set  $D_U$ , let  $x \in SU(2)$  and let  $f \in L^1(SU(2))$ . If  $f$  is continuous at each point of  $x D_U$  then the  $U$ -Fourier series for  $f$  does not converge deceptively at  $x$ . For each point  $y \in x D_U$  such that  $y \neq x$  there is a function in  $L^1(SU(2))$  which is analytic except at  $y$ , and whose  $U$ -Fourier series converges deceptively at  $x$ .*

**Proof.** Let  $j$  be an integer such that  $1 \leq j \leq [\frac{1}{2}(p-1)]$  and  $\cos((q-1)j\pi/p) \neq 0$ , and let  $K_j = \{x \in SU(2) : \theta(x) = 2\pi j/p\}$ . It is easy to verify that  $K_j \subseteq D_U$ . Suppose now that  $f$  is continuous at each point of  $x D_U$ . Then  ${}_x f$  is continuous at each point of  $D_U$  so for any  $\varepsilon > 0$  and any  $g \in K_j$  there exists a  $\delta(g) > 0$  such that

$$(7.3) \quad d(y, g) < \delta(g) \Rightarrow |{}_x f(y) - {}_x f(g)| < \frac{1}{2}\varepsilon.$$

The family of sets  $\{B(g, \delta(g)) : g \in K_j\}$  form an open cover for  $K_j$ . Let  $\delta$  be the Lebesgue number for this cover. Let  $g \in K_j$  and let  $h$  be any element of  $SU(2)$  such that  $d(g, h) < \delta$ . There exists a  $g_0 \in K_j$  such that  $g$  and  $h$  are both contained in  $B(g_0, \delta(g_0))$ , by the definition of Lebesgue number. Thus

$$(7.4) \quad |{}_x f(g) - {}_x f(h)| \leq |{}_x f(g) - {}_x f(g_0)| + |{}_x f(g_0) - {}_x f(h)| < \varepsilon$$

by (7.3). Now let  $s, t \in [0, \pi]$ . If  $|s - t| < \frac{1}{2}\delta$  then for all  $y \in SU(2)$

$$(7.5) \quad d(yx(s)y^{-1}, yx(t)y^{-1}) = d(x(s), x(t)) = 8^{1/2}|\sin \frac{1}{2}(s-t)| < \delta.$$



Now  $yx(2\pi j/p)y^{-1} \in K_j$  for all  $y \in SU(2)$ , and hence if  $|s - (2\pi j/p)| < \frac{1}{2}\delta$  it follows from (7.5) and (7.4) that

$$(7.6) \quad |{}_xf(yx(s)y^{-1}) - {}_xf(yx(2\pi j/p)y^{-1})| < \varepsilon.$$

Using the definition (5.6) of  $Q$  together with (7.6) we get

$$|[Q_x f](s) - [Q_x f](2\pi j/p)| < \varepsilon \quad \text{whenever } |s - (2\pi j/p)| < \delta/2,$$

and we have shown that  $[Q_x f]$  is continuous at  $2\pi j/p$  for all  $j$  such that  $1 \leq j \leq [\frac{1}{2}(p-1)]$  and  $\cos((q-1)\pi j/p) \neq 0$ . Thus it follows from Theorem 5.63 that if the  $U$ -Fourier series for  $f$  converges at  $x$  it must converge to  $f(x)$ . (Note that if  $p$  is even, then  $-e \in D_U$  so  ${}_xf$  is continuous at  $-e$ , and hence  $[Q_x f] \in L^1[-\pi, \pi]$ .) Now let

$$(7.7) \quad f = (1 - A)^{-1}$$

where  $A$  is defined by (6.1). It is shown in [5, p. 670] that  $f \in L^1(SU(2))$ , and an argument almost identical with an argument given there shows that the sequence  $\{f_n\}$  defined by

$$(7.8) \quad f_n = \sum_{j=0}^n A^j$$

converges to  $f$  in  $L^1(SU(2))$ . In [9, p. 164] it is shown that  $A^{j-1}$  is a coordinate function of the  $j$ -dimensional irreducible representation of  $SU(2)$ . Hence  $n\chi_n * A^{j-1} = \delta_{nj}A^{j-1}$ , and it follows that  $n\chi_n * f_j = A^{n-1}$  for all  $j \geq n-1$ . Thus  $n\chi_n * f = \lim_{j \rightarrow \infty} n\chi_n * f_j = A^{n-1}$  and

$$(7.9) \quad D_n * f = \sum_{j=1}^n j\chi_j * f = f_{n-1}.$$

If  $g$  is any element of  $SU(2)$  we can write

$$(7.10) \quad g = u^{-1}x(\theta(g))u$$

for some  $u \in SU(2)$ . Equation (7.10) does not determine  $u$  uniquely, but we will choose some  $u$  and keep it fixed in this discussion. If  $g \in SU(2)$  we define a function  $g^*$  in  $L^1(SU(2))$  by

$$(7.11) \quad g^*(y) = f(uyu^{-1}x(-\theta(g)))$$

where  $f$  is defined by (7.7).  $g^*$  is analytic except when  $uyu^{-1}x(-\theta(g)) = e$ , i.e. except when  $y = g$ . Also

$$(7.12) \quad g^*(e) = f(x(-\theta(g))) = (1 - \exp(-i\theta(g)))^{-1},$$

so that  $g^*(e)$  is finite for any  $g \neq e$ , and  $g^*(e) \neq 0$  for all  $g \in SU(2)$ . From now on we assume that  $g \neq e$ . For any  $y \in SU(2)$ ,  $D_n * g^*(y) = f_{n-1}(uyu^{-1}x(-\theta(g)))$ , and in particular, from (7.8) and (7.12) we have

$$D_n * g^*(e) = f_{n-1}(x(-\theta(g))) = g^*(e)[1 - \exp(-in\theta(g))].$$

Using this result together with (2.1) and (3.8) we obtain

$$(7.13) \quad U_n g^*(e) = g^*(e) - E_{qp}(g) \exp(-i(n-1)p\theta(g))$$

where

$$(7.14) \quad \begin{aligned} E_{qp}(g) &= g^*(e)(-1)^{q+1}(p-q+1) && \text{if } \theta(g) = \pi \\ &= g^*(e) \frac{e^{-i(q+1)\theta(g)}(1 + e^{-i(p-q+1)\theta(g)})}{1 + e^{-i\theta(g)}} && \text{if } \theta(g) \neq \pi. \end{aligned}$$

Suppose now that the eigenvalues of  $g$  are  $p$ th roots of unity, so that  $\theta(g) = 2\pi j/p$  for some integer  $j$ . Then  $\exp(-i(n-1)p\theta(g)) = 1$  for all  $n$ , and it follows from (7.13) that

$$(7.15) \quad \lim_{n \rightarrow \infty} U_n g^*(e) = g^*(e) - E_{qp}(g).$$

If  $\theta(g) = \pi$  (i.e. if  $g = -e$ ) then it follows from (7.14) that  $E_{qp}(g) \neq 0$ , and (7.15) shows that the  $U$ -Fourier series for  $g^*$  converges deceptively at  $e$ . If  $\theta(g) \neq \pi$  then (7.14) shows that  $E_{qp}(g) \neq 0$  if and only if  $\exp(-i(p-q+1)\theta(g)) \neq -1$ . It follows that the  $U$ -Fourier series for  $g^*$  converges deceptively at  $e$  whenever  $g$  is in the deception set  $D_U$  (and  $g \neq e$ ). If  $f$  is any function on  $SU(2)$ , and  $x \in SU(2)$  let  $f_x$  be the function on  $SU(2)$  defined by  $f_x(g) = f(x^{-1}g)$  for all  $g \in SU(2)$ . Let  $x \in SU(2)$  and let  $y = xz$  be a point in  $x D_U$  such that  $y \neq x$ . Then  $(z^*)_x$  is a function in  $L^1(SU(2))$  that is analytic except at  $y$ , and  $(z^*)_x(x) = z^*(e)$ , but the  $U$ -Fourier series for  $(z^*)_x$  at  $x$  converges to  $z^*(e) - E_{qp}(z) \neq z^*(e)$ . This completes the proof of Theorem 7.2.

**COROLLARY 7.16.** *Let  $U$  and  $V$  be two faithful representations of  $SU(2)$  which are series equivalent. Then  $U$  and  $V$  have the same deception sets.*

**Proof.** Suppose  $D_U \neq D_V$ , and let  $g$  be an element of  $SU(2)$  which is in exactly one of the sets  $D_U, D_V$ . Say  $g \in D_U, g \notin D_V$ . If  $g^*$  is constructed as in (7.11) then the  $U$ -Fourier series for  $g^*$  converges deceptively at  $e$ , but since  $g^*$  is analytic on  $D_V$  the  $V$ -Fourier series for  $g^*$  does not converge deceptively at  $e$ . Thus  $U$  and  $V$  are not series equivalent.

**COROLLARY 7.17.** *Let  $U$  be a faithful representation of  $SU(2)$ . Then  $U$ -Fourier series are honest for functions in  $L^1(SU(2))$  if and only if  $U$  is of type  $(1, 1)$ .*

**Proof.** Theorem 7.2 shows that  $U$ -Fourier series are honest for functions in  $L^1(SU(2))$  if and only if  $D_U = \{e\}$ . It follows from the definition of  $D_U$  that  $-e \in D_U$  if  $p$  is even,  $D_U$  contains all elements of  $SU(2)$  whose eigenvalues are  $p$ th roots of unity if  $p$  is odd, and  $D_U = \{e\}$  if and only if  $U$  is of type  $(1, 1)$ .

**PROPOSITION 7.18.** *Two faithful representations of  $SU(2)$  are of the same type if and only if they are series equivalent.*

**Proof.** By Corollary 3.19 we know that representations of the same type are series equivalent. Let  $U$  be a faithful representation of type  $(q, p)$  and let  $V$  be a faithful representation of type  $(r, s)$  and suppose that  $(q, p) \neq (r, s)$ . First consider

the case where  $p=s$ . Then  $q \neq r$ , so  $p > 2$ . Let  $g=x(2\pi/p)$ , so  $\theta(g)=2\pi/p \neq \pi$ . It follows from (7.14) and (7.15) that  $\lim_{n \rightarrow \infty} U_n g^*(e)$  and  $\lim_{n \rightarrow \infty} V_n g^*(e)$  both exist, and

$$(7.19) \quad \lim_{n \rightarrow \infty} U_n g^*(e) - \lim_{n \rightarrow \infty} V_n g^*(e) \\ = g^*(e)(\exp(2\pi i/p) + 1)^{-1}(\exp(-2\pi i r/p) - \exp(-2\pi i q/p)).$$

Since  $1 \leq r, q \leq p$  and  $q \neq r$  the right side of (7.19) is not zero, and hence  $U$  and  $V$  are not series equivalent if  $p=s$ . Now suppose that  $p \neq s$ . Say, for example  $s < p$  so  $p \geq 2$ . Let  $g=x(2\pi/p)$ . Then from definition (7.1) we see that  $g \notin D(r, s)$ , and  $g \in D(q, p)$  unless  $p$  is even and  $q-1 = \frac{1}{2}p$ . Thus by Corollary 7.16,  $U$  and  $V$  are not series equivalent if  $q-1 \neq \frac{1}{2}p$ . Suppose now that  $q-1 = \frac{1}{2}p$ . If  $s$  is odd we have  $-e \in D(q, p)$  and  $-e \notin D(r, s)$  and by Corollary 7.16  $U$  and  $V$  are not series equivalent. Suppose therefore, that  $s$  is even. Since  $p > s$ , we have  $p \geq 4$ , and since  $p=2(q-1)$  where  $q$  is even we have actually  $p \geq 6$ . Let  $h=x(4\pi/p) \neq -e$ . Then  $h \in D(q, p)$  but  $h \notin D(r, s)$  unless  $s = \frac{1}{2}p$ . It follows that  $U$  and  $V$  are not series equivalent unless possibly  $(q-1) = \frac{1}{2}p = s$ . If  $q-1 = \frac{1}{2}p = s$  then it follows from (7.14) and (7.15) that  $\lim_{n \rightarrow \infty} U_n h^*(e)$  and  $\lim_{n \rightarrow \infty} V_n h^*(e)$  both exist and

$$(7.20) \quad \lim_{n \rightarrow \infty} U_n h^*(e) - \lim_{n \rightarrow \infty} V_n h^*(e) \\ = h^*(e) \exp(-8\pi i/p)(1 + \exp(-4\pi i/p))^{-1}(\exp(-4(r-1)\pi i/p) - 1).$$

Since our hypotheses imply that  $0 < 2(r-1)/p < 1$  the right side of equation (7.20) is not zero, and  $U$  and  $V$  are again not series equivalent. Proposition 7.18 now follows.

**8. Deceptive convergence for  $L^2$  functions.** If  $f$  is a function in  $L^1(SU(2))$  then the *Riemann Lebesgue set* of  $f$  is defined to be

$$r(f) = \left\{ x \in SU(2) : \lim_{n \rightarrow \infty} n\chi_n * f(x) = 0 \right\}.$$

It follows from [6, Lemma 3] that  $\mu(r(f))=1$  for any  $f \in L^2(SU(2))$ .

**THEOREM 8.1.** *Let  $U$  be a faithful representation of  $SU(2)$ , and let  $f$  be a function in  $L^1(SU(2))$ . Then the  $U$ -Fourier series for  $f$  does not converge deceptively at any point of the Riemann Lebesgue set of  $f$ . In particular, if  $f \in L^2(SU(2))$  then the set of points where the  $U$ -Fourier series for  $f$  converges deceptively has measure zero.*

**Proof.** Let  $x$  be a point of continuity of  $f$  which is contained in  $r(f)$ , and suppose that the  $U$ -Fourier series for  $f$  at  $x$  converges to  $L$ . Let  $U$  be of type  $(q, p)$  and let  $V$  be a representation of  $SU(2)$  of type  $(1, 1)$ . For any positive integer  $n$  let  $m$  be the largest integer such that  $(m-1)p+q+1 \leq n$ . Then  $m \rightarrow \infty$  as  $n \rightarrow \infty$ . By (3.18) and the relationship  $D_r - D_{r-1} = r\chi_r$ , we have

$$(8.2) \quad V_n f(x) - U_m f(x) = \sum_{j=(m-1)p+q+2}^{n+1} (j\chi_j * f)(x) \\ - \sum_{j=1}^{(p-q)/2} ((m-1)p+q+2j+1)\chi_{(m-1)p+q+2j+1} * f(x).$$

By the definition of  $m$  we have  $0 \leq (n+1) - ((m-1)p + q + 2) < p$ . Since

$$\lim_{j \rightarrow \infty} (j\chi_j * f)(x) = 0$$

it follows from (8.2) that  $\lim_{n \rightarrow \infty} V_n f(x) - U_m f(x) = 0$ , and hence

$$\lim_{n \rightarrow \infty} V_n f(x) = \lim_{m \rightarrow \infty} U_m f(x) = L.$$

By Corollary 7.17 we have  $L = f(x)$ , and the  $U$ -Fourier series for  $f$  does not converge deceptively at  $x$ .

#### BIBLIOGRAPHY

1. N. Bourbaki, *Intégration*, Actualités Sci. Indust, no. 1306, Hermann, Paris, 1963.
2. C. Chevalley, *Theory of Lie groups*, Princeton Univ. Press, Princeton, N. J., 1946.
3. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Univ. Press, Princeton, N. J., 1941.
4. L. Loomis, *Abstract harmonic analysis*, Van Nostrand, New York, 1953.
5. R. Mayer, *Summation of Fourier series on compact groups*, Amer. J. Math. **89** (1967), 661–692.
6. ———, *Fourier series of differentiable functions on  $SU(2)$* , Duke Math. J. **34** (1967), 549–554.
7. G. Mackey, *Infinite-dimensional group representations*, Bull. Amer. Math. Soc. **69** (1963), 628–686.
8. H. Weyl, *Gruppentheorie und Quantenmechanik*, Hirzel, Leipzig; English transl., Dover, New York, 1950.
9. E. P. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atom-spektren*, J. W. Edwards, Ann Arbor, Mich.; English transl., Academic Press, New York, 1959.
10. A. Zygmund, *Trigonometric series*. Vol. I, Monografie Mat., Bd. 5, PWN, Warsaw, 1935; 2nd rev. ed., Cambridge Univ. Press, New York, 1959.

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